STABLE STRONG AND TOTAL PARAMETRIZED DUALITIES FOR DC OPTIMIZATION PROBLEMS IN LOCALLY CONVEX SPACES

GANG LI
Harbin Institute of Technology Shenzhen Graduate School
Shenzhen 518055, China

XIAOQI YANG
Department of Applied Mathematics
The Hong Kong Polytechnic University
Hung Hum, Kowloon, Hong Kong, China

YUYING ZHOU
Department of Mathematics, Soochow University
Suzhou 215006, China

(Communicated by Adil Bagirov)

Abstract. By using properties of dualizing parametrization functions, Lagrangian functions and the epigraph technique, some sufficient and necessary conditions of the stable strong duality and strong total duality for a class of DC optimization problems are established.

1. Introduction. Duality theory is an elegant and powerful tool in optimization. Many primal problems can be studied via their dual problems. A challenge in convex programming has been to give necessary and sufficient conditions which guarantee the strong duality, that is, the situation when the values of the primal problem and the dual problem coincide and the dual problem has at least an optimal solution. For this purpose, the Slater condition was used in [8] and various interior-type conditions were established in [1, 2, 11]. The epigraph technique has also been used extensively and shown great power in deriving new constraint qualification conditions, which are normally weaker than the interior-type conditions (cf. [5, 6, 7]).

Some related and interesting problems are the total duality, which corresponds to the situation that the values of the primal problem and the dual problem coincide and both of them have optimal solutions, and the stable duality, where the usual duality results remain valid under the perturbation of linear functions. Some necessary and sufficient conditions for total Lagrange duality and total Fenchel-Lagrange duality were given for convex optimization problems in [3] and [4]. Sufficient and necessary conditions were obtained for the stable conjugate duality in [7, 13] and for the stable Fenchel duality and the total Fenchel duality in [15].

2010 Mathematics Subject Classification. Primary: 90C26, 90C46; Secondary: 90C30.

Key words and phrases. Conjugate functions, epigraph, DC programming.

*Corresponding author. The work was supported by the Research Grants Council of Hong Kong (PolyU 5306/11E) and Natural Science Foundation of China (10831009, 11071180 and 11171247).
In the recent years, the optimization problem with a difference of two convex functions (DC in short) in either the objective function or the constraints, or both, has received extensive attention. Similar to the convex duality theory, a constraint qualification is an essential ingredient for guaranteeing the strong duality. A conjugate duality theory for optimization problems with a DC objective function and finitely many DC constraints was established in [14, 17] under a Slater condition. A closed condition involving DC objective functions and a convex constraint system was given in [9]. In [12], characterizations of global optimality for general difference convex (DC) optimization problems involving convex inequality constraints were given in terms of \(\epsilon\)-subdifferentials. In [19], a sequential convex program method was constructed to solve a DC programming problem with joint chance constraints. Some necessary and sufficient conditions of the strong Fenchel duality, the stable Fenchel duality and the stable total duality for DC programs were given in [10].

Let \(X\) and \(Y\) be locally convex Hausdorff topological vector spaces. In this paper, we will consider the following DC problem:

\[
(P_A) \quad \inf_{x \in X} \{ f_1(x) - f_2(x) + g_1(Ax) - g_2(Ax) \},
\]

where \(f_1, f_2 : X \to \mathbb{R} := \mathbb{R} \cup \{\pm \infty\}\) and \(g_1, g_2 : Y \to \mathbb{R}\) are proper convex functions, and \(A : X \to Y\) is a linear continuous operator. The dualizing parametrization technique was used in [18] to study the duality theory of convex programming. This technique has recently been employed in the study of stable conjugate duality theory of convex programming in [7]. In this paper, we will formulate a type of dual problems of \((P_A)\) by using dualizing parametrization functions and Lagrangian functions. By virtue of the epigraph technique, we will establish some sufficient and necessary conditions of the stable strong duality and stable total duality.

The paper is organized as follows. In Section 2, we introduce some notations and recall some preliminary results needed in the sequel of this article. In Section 3, we introduce parametrized Lagrange dual problems. We consider the stable strong dualities and the stable total dualities in Sections 4 and 5, respectively.

2. Preliminaries. First, we will give some notations and preliminaries. The notation used in here is standard (cf. [18, 20]). We assume throughout the whole paper that \(X\) and \(Y\) are locally convex Hausdorff topological vector spaces, \(X^*\) and \(Y^*\) are the dual spaces of \(X\) and \(Y\), respectively.

Let \(U\) be a set in \(X\). The closure of \(U\) is denoted by \(\text{cl} \ U\). If \(W \subseteq X^*\), then \(\text{cl}^* \ W\) denotes the weak*-closure of \(W\) and \(\text{co} \ W\) denotes the convex hull. We endow \(X^* \times \mathbb{R}\) with product topology of \(w^*(X^*, X)\) and the usual Euclidean topology. By \(\langle x^*, x \rangle\) we shall denote the value of the functional \(x^* \in X^*\) at \(x \in X\), i.e., \(\langle x^*, x \rangle = x^*(x)\). Let \(A^* : Y^* \to X^*\) be the adjoint operator defined by \(A^* y^*(x) = y^*(Ax)\) for all \((x, y^*) \in X \times Y^*\).

The indicator function \(\delta_U\) and the support function \(\sigma_U\) of the nonempty set \(U\) are respectively defined by

\[
\delta_U(x) := \begin{cases} 
0, & \text{if } x \in U, \\
+\infty, & \text{otherwise}, 
\end{cases}
\]

and

\[
\sigma_U(x^*) := \sup_{x \in U} x^*(x), \text{ for each } x^* \in X^*.
\]
Let $f : X \to \overline{\mathbb{R}}$ be a convex function. The effective domain and the epigraph of $f$ are respectively defined by

$$\text{dom } f := \{ x \in X : f(x) < +\infty \},$$

and

$$\text{epi } f := \{(x,r) \in X \times \mathbb{R} : f(x) \leq r \}.$$

The function $f$ is said to be proper if $f$ does not take on the value $-\infty$ and $\text{dom } f \neq \emptyset$. The conjugate function of $f$ is the function $f^* : X^* \to \overline{\mathbb{R}}$ defined by

$$f^*(x^*) := \sup \{ x^*(x) - f(x) : x \in X \}.$$

The function $f$ is lower-semicontinuous (l.s.c.) if the set $\{ x : f(x) \leq \alpha \}$ is closed for all $\alpha \in \mathbb{R}$. This is equivalent to the condition that $\text{epi } f$ is closed as a subset of $X \times \mathbb{R}$. The l.s.c. hull of $f$, denoted by $\text{lsc } f$, satisfies

$$\text{epi}(\text{lsc } f) = \text{cl}(\text{epi } f).$$

Define the closure of $f$ by

$$\text{cl } f(x) = \begin{cases} \text{lsc } f(x), & \text{if } x \in X, \text{ lsc } f(x) > -\infty, \\ -\infty, & \text{otherwise}. \end{cases}$$

We say $f$ is closed if $f = \text{cl } f$, i.e., if $f$ is a l.s.c. function and does not take the value $-\infty$, or if $f$ is the constant function $-\infty$.

If $f$ is convex, then (cf. [20, Theorem 2.3.1]),

$$f^* = (\text{cl } f)^*.$$  \hfill (1)

By [20, Theorem 2.3.3], if $f$ is proper, l.s.c. and convex, then the following equality holds:

$$f^{**} = f.$$  \hfill (2)

Let $x \in X$. The subdifferential of $f$ at $x$ is defined by

$$\partial f(x) := \{ x^* \in X^* : f(x) + x^*(y-x) \leq f(y) \text{ for each } y \in X \}$$

if $x \in \text{dom } f$, and $\partial f(x) := \emptyset$ otherwise. By [20, Theorems 2.3.1 and 2.4.2(ii)], the Young-Fenchel inequality below holds:

$$f(x) + f^*(x^*) \geq x^*(x) \text{ for each } x \in X, x^* \in X^*,$$

where the equality holds if and only if $x^* \in \partial f(x)$.

If $g, h : X \to \overline{\mathbb{R}}$ are proper, then

$$\text{epi } g^* + \text{epi } h^* \subseteq \text{epi } (g + h)^*,$$

$$g \leq h \Rightarrow g^* \geq h^* \iff \text{epi } g^* \subseteq \text{epi } h^*,$$

and

$$\partial g(x) + \partial h(x) \subseteq \partial(g + h)(x) \text{ for each } x \in \text{dom } g \cap \text{dom } h.$$  \hfill (3)

Furthermore, if $g, h$ are convex and $\text{cl } g, \text{ cl } h$ are proper, then

$$\text{cl } g \leq \text{cl } h \iff \text{epi } g^* \subseteq \text{epi } h^*.$$
For each \( a \in \mathbb{R}, \ p \in X^* \) and any proper function \( h : X \to \overline{\mathbb{R}}, \) let \( H(x) = h(x) + a, \) we have

\[
(x^*, r) \in \text{epi}(h + p + a)^* \Leftrightarrow (x^*, r) \in \text{epi}(H + p)^*
\]

\[
\Leftrightarrow \sup_{x \in X} \{x^*(x) - (H + p)(x)\} \leq r
\]

\[
\Leftrightarrow \sup_{x \in X} \{(x^* - p)(x) - h(x)\} \leq r + a
\]

\[
\Leftrightarrow (x^* - p, r + a) \in \text{epi} h^*
\]

So we have

\[
\text{epi} (h + p + a)^* = \text{epi} h^* + (p, -a).
\]

Moreover, throughout the whole paper, we assume that

\[
D := \text{dom}(f_1 - f_2) \cap \text{dom}((g_1 - g_2) \circ A) \neq \emptyset.
\]

Following [20], we adopt the convention that \((+\infty) + (-\infty) = (+\infty) - (+\infty) = +\infty \) and \(0 \cdot \infty = 0.\)

### 3. Parametrized Lagrange dual problems

Consider the problem \((P_A).\) We define the dualizing parametrization functions \(F_i : X \times Z \to \overline{\mathbb{R}}, \) \(G_i : Y \times Z \to \overline{\mathbb{R}}, i = 1, 2\) for \(f_i(x)\) and \(g_i(y)\) respectively by

\[
F_i(x, 0) = f_i(x), \quad G_i(y, 0) = g_i(y).
\]

where \(Z\) is also a locally convex Hausdorff topological vector space and \(Z^*\) is the dual space. Assume throughout the paper that \(F_i, G_i, (i = 1, 2)\) are proper convex functions and \(F_i(x, \cdot), G_i(y, \cdot)(i = 1, 2)\) are closed. Everything below depends on the particular choice of \(F_i\) and \(G_i\) (see [18] for more details).

Similar to the methods used on pages 18 – 19 in [18], for problem \((P_A),\) we define the Lagrangian function \(K_i\) on \(X \times Z^*\) and \(L_i\) on \(Y \times Z^*(i = 1, 2)\) by

\[
K_i(x, z^*) = \inf_{z \in Z} \{F_i(x, z) + z^*(z)\},
\]

\[
L_i(Ax, z^*) = \inf_{z \in Z} \{G_i(Ax, z) + z^*(z)\}.
\]

Thus the functions \(z^* \to K_i(x, z^*)\) and \(z^* \to L_i(Ax, z^*)\) are the conjugates in the concave sense of the functions \(z \to -F_i(x, z)\) and \(z \to -G_i(Ax, z),\) respectively. Since \(F_i(x, \cdot), G_i(y, \cdot)(i = 1, 2)\) are closed and convex, the conjugates are reciprocal, i.e., the functions \(z \to -F_i(x, z)\) and \(z \to -G_i(Ax, z)\) are the conjugates of the functions \(z^* \to K_i(x, z^*)\) and \(z^* \to L_i(Ax, z^*),\) or

\[
F_i(x, z) = \sup_{z^* \in Z^*} \{K_i(x, z^*) - z^*(z)\},
\]

\[
G_i(Ax, z) = \sup_{z^* \in Z^*} \{L_i(Ax, z^*) - z^*(z)\}.
\]

By (7), (10) and (11), we obtain

\[
f_i(x) = \sup_{z^* \in Z^*} K_i(x, z^*), \quad g_i(Ax) = \sup_{z^* \in Z^*} L_i(Ax, z^*), \quad i = 1, 2.
\]

For given \(z^* \in Z^*,\) denote

\[
K_i^*(z^*) = K_i(x, z^*), \quad L_i^*(Ax) = L_i(Ax, z^*), \quad i = 1, 2.
\]

To properly formulate a dual problem for \((P_A),\) we further assume throughout the paper that there exist some \(z^* \in Z^*,\) such that \(K_i^*\) and \(L_i^*\) are proper l.s.c.
functions. By Theorem 6 in [18], \(K_i^+(x)\) and \(L_i^+(y)\) are convex. It follows from (2) that

\[
K_2^+(x) = (K_2^+)^*(x) = \sup_{u^* \in \text{dom}(K_2^+)^*} \{u^*(x) - (K_2^+)^*(u^*)\}, 
\]

\[
L_2^+(Ax) = (L_2^+)^*(Ax) = \sup_{v^* \in \text{dom}(L_2^+)^*} \{v^*(Ax) - (L_2^+)^*(v^*)\}. 
\]

It follows from (14) and (15) that

\[
\inf_{x \in X} \{ K_1(x, z^*) - K_2(x, z^*) + L_1(Ax, z^*) - L_2(Ax, z^*) \} 
= \inf_{x \in X, (u^*, v^*) \in H^*_x} \{ K_1^+(x) - u^*(x) + (K_2^+)^*(u^*) + L_1^+(Ax) - v^*(Ax) + (L_2^+)^*(v^*) \} 
= \inf_{(u^*, v^*) \in H^*_x} \{ - (K_1^+ + L_1^+ \circ A)^*(u^* + A^* v^*) + (K_2^+)^*(u^*) + (L_2^+)^*(v^*) \},
\]

where \(L_1^+ \circ A : X \to \mathbb{R}, (L_1^+ \circ A)(x) = L_1^+(Ax)\) and

\[
H^*_x = \text{dom}(K_2^+)^* \times \text{dom}(L_2^+)^* \neq \emptyset.
\]

Define

\[
\psi(z^*) = \inf_{(u^*, v^*) \in H^*_x} \{ - (K_1^+ + L_1^+ \circ A)^*(u^* + A^* v^*) + (K_2^+)^*(u^*) + (L_2^+)^*(v^*) \}.
\]

From (16)–(18), we define the following dual problem of \((P_A)\):

\[
(D_A) \quad \sup_{z^* \in Z^*} \psi(z^*).
\]

One of interesting problems in recent years in the community is the study of stable dualities between \((P_A)\) and \((D_A)\) (see e.g. [15]). That is, the strong duality or the total duality holds between a linearly perturbed problem of \((P_A)\) and its dual. Given \(p \in X^*,\) consider the following DC problem with a linear perturbation

\[
(P_{(A,p)}) \quad \inf_{x \in X} \{ f_1(x) - f_2(x) + g_1(Ax) - g_2(Ax) - p(x) \},
\]

and the corresponding dual problem

\[
(D_{(A,p)}) \quad \sup_{z^* \in Z^*} \inf_{(u^*, v^*) \in H^*_x} \{ - (K_1^+ + L_1^+ \circ A)^*(u^* + p + A^* v^*) + (K_2^+)^*(u^*) + (L_2^+)^*(v^*) \}.
\]

Let \(v(P_A), v(P_{(A,p)}), v(D_A)\) and \(v(D_{(A,p)})\) denote the optimal values of problems \((P_A), (P_{(A,p)}), (D_A)\) and \((D_{(A,p)})\), respectively.

The zero duality gap between \((P_A)\) and \((D_A)\) states that

\[
v(P_A) = v(D_A),
\]

the strong duality between \((P_A)\) and \((D_A)\) states that

\[
v(P_A) = v(D_A) \quad \text{and the problem} \quad (D_A) \quad \text{has an optimal solution},
\]

the total duality between \((P_A)\) and \((D_A)\) states that

\[
v(P_A) = v(D_A) \quad \text{and both problems} \quad (P_A) \quad \text{and} \quad (D_A) \quad \text{have optimal solutions, and}
\]

the stable strong duality between \((P_A)\) and \((D_A)\) states that, for each \(p \in X^*,\)

\[
v(P_{(A,p)}) = v(D_{(A,p)}) \quad \text{and the problem} \quad (D_{(A,p)}) \quad \text{has an optimal solution}.
\]
Unlike the convex case, the weak duality between \((P_{(A,p)})\) and \((D_{(A,p)})\) (i.e., \(v(P_{(A,p)}) \geq v(D_{(A,p)})\)) does not hold, in general, as to be shown in Example 2 in this section.

**Lemma 3.1.** (Theorem 2.1.3 (i) [20]) Let I be an index set and let \(\{h_i : i \in I\}\) be a family of proper convex functions. Then \(\text{epi}(\text{sup}_{i \in I} h_i) = \cap_{i \in I} \text{epi} h_i\).

**Lemma 3.2.** (Lemma 2.2 and Lemma 2.3 [16]) Let I be an index set and let \(\{h_i : i \in I\}\) be a family of proper l.s.c. convex functions on \(X\) with \(\sup_{i \in I} h_i(x_0) < +\infty\) for some \(x_0 \in X\). Then \((\sup_{i \in I} h_i)^* = \text{cl}^* \text{co}(\inf_{i \in I} h_i^*)\) and \(\text{epi}(\sup_{i \in I} h_i)^* = \text{cl}^* \text{co}(\cup_{i \in I} \text{epi} h_i^*)\).

We define the following characteristic set \(\Lambda\),

\[
\Lambda := \bigcup_{z^* \in Z^*} \left( \bigcap_{(u^*,v^*) \in H^*_z} \{ \text{epi}(K_1^* + L_1^* \circ A)^* \}ight)
\]

\[
- (u^*, (K_2^*)^*(v^*)) - (A^* v^*, (L_2^*)^*(v^*)) \right) \right].
\]

In fact, \(\Lambda\) has an alternative expression as given in the following lemma.

**Lemma 3.3.** Assume that \(F_i, G_i\) \((i = 1, 2)\) are proper convex functions, \(F_i(x, \cdot)\), \(G_i(y, \cdot)\) are closed functions and there exist some \(z^* \in Z^*\), such that \(K_2^*, L_2^*\) are proper l.s.c. functions. Then we have the following formula:

\[
\Lambda = \bigcup_{z^* \in Z^*} \text{epi}(K_1^* - K_2^* + L_1^* \circ A - L_2^* \circ A)^*.
\]

**Proof.** Using (14) and (15), one gets that

\[
(K_1^* - K_2^* + L_1^* \circ A - L_2^* \circ A)^*(x^*)
\]

\[
= \sup_{x \in X, (u^*,v^*) \in H^*_z} \{ x^*(x) - K_1^*(x) + u^*(x) - (K_2^*)^*(v^*) 
\]

\[
- L_1^*(Ax) + v^*(Ax) - (L_2^*)^*(v^*) \}
\]

\[
= \sup_{(u^*,v^*) \in H^*_z} \{ (K_1^* - u^* + (K_2^*)^*(v^*)) + L_1^* \circ A - A^* v^* + (L_2^*)^*(v^*))^*(x^*). \}
\]

This, together with the Lemma 3.1 and (5), implies that

\[
\text{epi}(K_1^* - K_2^* + L_1^* \circ A - L_2^* \circ A)^*
\]

\[
= \bigcap_{(u^*,v^*) \in H^*_z} \{ \text{epi}(K_1^* - u^* + (K_2^*)^*(v^*)) + L_1^* \circ A - A^* v^* + (L_2^*)^*(v^*))^* \}
\]

\[
= \bigcap_{(u^*,v^*) \in H^*_z} \{ \text{epi}(K_1^* + L_1^* \circ A)^* - (u^*, (K_2^*)^*(v^*)) - (A^* v^*, (L_2^*)^*(v^*)) \} \}
\]

So, (20) holds.

**Remark 1.** Comparing with the expressions of problems \((P_{(A,p)})\) and \((D_{(A,p)})\), we know that the set \(\Lambda\) is associated with \((D_{(A,p)})\) and \(\text{epi}(f_1 - f_2 + g_1 \circ A - g_2 \circ A)^*\) is associated with \((P_{(A,p)})\). But the inclusion relations between \(\Lambda\) and \(\text{epi}(f_1 - f_2 + g_1 \circ A - g_2 \circ A)^*\) are not necessarily true.
Example 1. Let $X = Y = Z = \mathbb{R}$ and $A$ be the identity operator. Define $f_1, f_2, g_1, g_2 : \mathbb{R} \to \mathbb{R}$ respectively by $f_1 = \delta_{[-2, +\infty)}, f_2 = g_1 = \delta_{[1, +\infty)}$, 

$$g_2(x) = \begin{cases} -2x, & \text{if } x \geq 1, \\ +\infty, & \text{otherwise}, \end{cases}$$

where $\delta_{[-2, +\infty)}, \delta_{[1, +\infty)}$ are the indicator functions. Then, $f_1, f_2, g_1$ and $g_2$ are proper convex functions. Define 

$$F_i(x, z) = \begin{cases} 0, & \text{if } x \geq -2 \text{ and } z \leq 2 + x, \\ +\infty, & \text{otherwise}, \end{cases}$$

$$F_2(x, z) = \begin{cases} 0, & \text{if } x \geq 1 \text{ and } z \leq 0.5 + 2x, \\ +\infty, & \text{otherwise}, \end{cases}$$

and $G_i(Ax, z) = g_i(Ax) + z$, $i = 1, 2$. Then

$$K_1(x, z^*) = \begin{cases} z^*(2 + x), & \text{if } x \geq -2, z^* \leq 0, \\ -\infty, & \text{if } x \geq -2, z^* > 0, \\ +\infty, & \text{otherwise}, \end{cases}$$

$$L_1(Ax, z^*) = \begin{cases} 0, & \text{if } x \geq 1, z^* = -1, \\ -\infty, & \text{if } x \geq 1, z^* \neq -1, \\ +\infty, & \text{otherwise}, \end{cases}$$

$$K_2(x, z^*) = \begin{cases} z^*(0.5 + 2x), & \text{if } x \geq 1, z^* \leq 0, \\ -\infty, & \text{if } x \geq 1, z^* > 0, \\ +\infty, & \text{otherwise}, \end{cases}$$

$$L_2(Ax, z^*) = \begin{cases} -2x, & \text{if } x \geq 1, z^* = -1, \\ -\infty, & \text{if } x \geq 1, z^* \neq -1, \\ +\infty, & \text{otherwise}. \end{cases}$$

Simple calculations give

$$\Lambda = \{(a, r) : a \leq 3, r \geq a - 1.5\},$$

$$\text{epi}(f_1 - f_2 + g_1 \circ A - g_2 \circ A)^* = \{(a, r) : a \leq 2, r \geq a - 2\}.$$ 

Therefore, $\Lambda \nsubseteq \text{epi}(f_1 - f_2 + g_1 \circ A - g_2 \circ A)^*$ and $\Lambda \nsubseteq \text{epi}(f_1 - f_2 + g_1 \circ A - g_2 \circ A)^*$.

Remark 2. The following example shows that, the weak duality between $(P_{(A, p)})$ and $(D_{(A, p)})$ does not hold in general.

Example 2. Let $X = Y = Z = \mathbb{R}$ and $p = 0$. $f_i, g_i, F_i, G_i, K_i$ and $L_i$ ($i = 1, 2$) are defined in Example 1. Then $v(P_A) = 2$ and $v(D_A) = +\infty$, which implies that $v(P_A) < v(D_A)$. The weak duality does not hold.

4. Stable strong dualities. Let $f_1, f_2 : X \to \mathbb{R}$ and $g_1, g_2 : Y \to \mathbb{R}$ be proper convex functions such that $f_1 - f_2, g_1 - g_2$ are proper functions and $\mathcal{D} \neq \emptyset$, where $\mathcal{D}$ is defined by (6). Assume that $F_i, G_i$ ($i = 1, 2$) are proper convex functions, such that $F_i(x, 0) = f_i(x), G_i(Ax, 0) = g_i(Ax)$. $F_i(x, \cdot), G_i(y, \cdot)$ are closed functions and there exist some $z^* \in Z^*$, such that $K_2^+, L_2^+$ are proper l.s.c. functions, where $K_2^+, L_2^+$ are defined by (13).

Consider the possible inclusions between $\text{epi}(f_1 - f_2 + g_1 \circ A - g_2 \circ A)^*$ and $\Lambda$. In order to study the weak duality and the strong duality, we introduce the following definitions.
Remark 3. Since $\Lambda \subseteq \text{cl}^* \Lambda$, each of $(CC)_p$ and $(ACC)_p$ implies $(SCC)_p$.

It is worth noting that since

$$v(P(A,p)) = \inf_{x \in \Lambda} \{f_1(x) - f_2(x) + g_1(Ax) - g_2(Ax) - p(x)\}$$

we have

$$v(P(A,p)) \geq -r \iff (p,r) \in \text{epi}(f_1 - f_2 + g_1 \circ A - g_2 \circ A)^*.$$  \hspace{1cm} (24)

Lemma 4.2. Let the set $\Lambda$ be defined by (19) and $r \in \mathbb{R}$. Then $(p,r) \in \Lambda$ if and only if there is $z^* \in Z^*$ such that each $(u^*, v^*) \in H_{z^*}^*$ satisfies

$$- (K_i^z + L_i^z \circ A)^*(u^* + p + A^*v^*) + (K_2^z)^*(u^*) + (L_2^z)^*(v^*) \geq -r,$$  \hspace{1cm} (25)

where $K_i^z(x), L_i^z(Ax) (i = 1, 2)$ are defined by (13) and $H_{z^*}^*$ is defined by (17).

Proof. $[\Rightarrow]$ Let $(p,r) \in \Lambda$. Then there is $z^* \in Z^*$, such that for each $(u^*, v^*) \in H_{z^*}^*$,

$$(p,r) \in \text{epi}(K_1^z + L_1^z \circ A)^* - (u^*, (K_2^z)^*(u^*)) - (A^*v^*, (L_2^z)^*(v^*)).$$

Thus, there exists $(x_1^*, r_1) \in \text{epi}(K_1^z + L_1^z \circ A)^*$ such that

$$x_1^* - u^* - A^*v^* = p, \quad r_1 - (K_2^z)^*(u^*) - (L_2^z)^*(v^*) = r,$$  \hspace{1cm} (26)

and

$$(K_i^z + L_i^z \circ A)^*(x_1^*) \leq r_1.$$  \hspace{1cm} (27)

So it follows from (26) and (27) that

$$- (K_1^z + L_1^z \circ A)^*(u^* + p + A^*v^*) + (K_2^z)^*(u^*) + (L_2^z)^*(v^*)$$

$$= - (K_1^z + L_1^z \circ A)^*(x_1^*) + (K_2^z)^*(u^*) + (L_2^z)^*(v^*)$$

$$\geq -r_1 + (K_2^z)^*(u^*) + (L_2^z)^*(v^*) = -r.$$

That is, (25) holds.

$[\Leftarrow]$ Suppose that there is $z_0^* \in Z^*$ such that each $(u^*, v^*) \in H_{z_0^*}^*$ satisfies (25). Then

$$(K_1^z + L_1^z \circ A)^*(u^* + p + A^*v^*) \leq r + (K_2^z)^*(u^*) + (L_2^z)^*(v^*),$$

that is,

$$(u^* + p + A^*v^*, r + (K_2^z)^*(u^*) + (L_2^z)^*(v^*)) \in \text{epi}(K_1^z + L_1^z \circ A)^*.$$  \hspace{1cm} (28)
It follows from (28) that,
\[
(p, r) = (u^* + p + A^*v^*, r + (K_{2,0}^{\ast})^*(u^*) + (L_{2,0}^{\ast})^*(v^*))
\]
\[-\left(u^*, (K_{2,0}^{\ast})^*(u^*) - (A^*v^*, (L_{2,0}^{\ast})^*(v^*))\right)
\in \text{epi}(K_{2,0}^{\ast} + L_{2,0}^{\ast} \circ A)^* - \left(u^*, (K_{2,0}^{\ast})^*(u^*) - (A^*v^*, (L_{2,0}^{\ast})^*(v^*))\right).
\]
Noting that \((u^*, v^*) \in H_{z_0}^*\) is arbitrary, we have
\[
(p, r) \in \bigcap_{(u^*, v^*) \in H_{z_0}^*} \{\text{epi}(K_1^{\ast} + L_1^{\ast} \circ A)^* - \left(u^*, (K_2^{\ast})^*(u^*) - (A^*v^*, (L_2^{\ast})^*(v^*))\right)\}.
\]
By (19), \((p, r) \in \Lambda\).

As shown in Example 2, the weak duality between \(P_{(A,p)}\) and \(D_{(A,p)}\) does not hold in general. In the following, we give a necessary and sufficient condition for the weak duality between \(P_{(A,p)}\) and \(D_{(A,p)}\).

**Theorem 4.3.** For each \(p \in X^*\), the weak duality between \((P_{(A,p)})\) and \((D_{(A,p)})\) holds if and only if the family \((f_1, f_2, g_1, g_2; A)\) satisfies \((SCC)_P\).

**Proof.** [\(\Rightarrow\)] Suppose the weak duality holds, that is, \(v(P_{(A,p)}) \geq v(D_{(A,p)})\). Let \((p, r) \in \Lambda\). By Lemma 4.2, \(v(D_{(A,p)}) \geq -r\), hence \(v(P_{(A,p)}) \geq -r\). By (24), we have
\[
(p, r) \in \text{epi}(f_1 - f_2 + g_1 \circ A - g_2 \circ A)^*.
\]
Hence \((SCC)_P\) holds.

[\(\Leftarrow\)] Suppose \((SCC)_P\) holds, i.e., (23) holds. Let \(r \leq v(D_{(A,p)})\). Recall that \(v(D_{(A,p)})\) is the optimal value of the problem \((D_{(A,p)})\). For arbitrary \(\varepsilon > 0\), there is \(z^* \in Z^*\), for each \((u^*, v^*) \in H_{z^*}^*\),
\[
-(K_1^{\ast} + L_1^{\ast} \circ A)^* (u^* + p + A^*v^*) + (K_2^{\ast})^*(u^*) + (L_2^{\ast})^*(v^*) \geq r - \varepsilon.
\]
By Lemma 4.2 and (23),
\[
(p, -r + \varepsilon) \in \Lambda \subseteq \text{epi}(f_1 - f_2 + g_1 \circ A - g_2 \circ A)^*.
\]
which implies that \(v(P_{(A,p)}) \geq r - \varepsilon\), thanks to (24). Consequently, we have \(v(P_{(A,p)}) \geq v(D_{(A,p)})\). Hence, the weak duality between \((P_{(A,p)})\) and \((D_{(A,p)})\) holds and the proof is complete.

**Proposition 1.** For each \(p \in X^*\), the zero duality gap between \((P_{(A,p)})\) and \((D_{(A,p)})\) holds if and only if the family \((f_1, f_2, g_1, g_2; A)\) satisfies \((ACC)_P\).

**Proof.** [\(\Rightarrow\)] Suppose \(v(P_{(A,p)}) = v(D_{(A,p)}), \ p \in X^*\). By Theorem 4.3, the family \((f_1, f_2, g_1, g_2; A)\) satisfies \((SCC)_P\), i.e., (23) holds. Therefore
\[
\text{cl}^* \Lambda \subseteq \text{cl}^* \text{epi}(f_1 - f_2 + g_1 \circ A - g_2 \circ A)^*.
\]
Let
\[
(x^*, s) \in \text{cl}^* \text{epi}(f_1 - f_2 + g_1 \circ A - g_2 \circ A)^*.
\]
Then there exists \(\{x^*_\alpha, s_\alpha\} \subseteq \text{epi}(f_1 - f_2 + g_1 \circ A - g_2 \circ A)^*, \) such that \((x^*_\alpha, s_\alpha) \to (x^*, s)\). Thus by (24), \(v(P_{(A,z^*_\alpha)}) \geq -s_\alpha\), hence \(v(D_{(A,z^*_\alpha)}) \geq -s_\alpha\), i.e.,
\[
\sup_{z^* \in Z^*} \inf_{(u^*, v^*) \in H_{z^*}^*} \{-(K_1^{\ast} + L_1^{\ast} \circ A)^* (u^* + x^*_\alpha + A^*v^*)
\]
\[+(K_2^{\ast})^*(u^*) + (L_2^{\ast})^*(v^*) \} \geq -s_\alpha.
\]
By (16), the above inequality can be rewritten as follows,
\[
\sup_{z^* \in Z^*} \inf_{x \in X} \{ K_1^z(x) - K_2^z(x) + L_1^z(Ax) - L_2^z(Ax) - x^*_\alpha(x) \} \geq -s_\alpha.
\]
Since \((x^*_\alpha, s_\alpha) \to (x^*, s)\),
\[
\sup_{z^* \in Z^*} \inf_{x \in X} \{ K_1^z(x) - K_2^z(x) + L_1^z(Ax) - L_2^z(Ax) - x^*(x) \} \geq -s,
\]
i.e., \(v(D_{(A,x^*)}) \geq -s\). Then \(v(P_{(A,x^*)}) \geq -s\). By (24), \((x^*, s) \in \text{epi}(f_1 - f_2 + g_1 \circ A - g_2 \circ A)^*\). So \(\text{epi}(f_1 - f_2 + g_1 \circ A - g_2 \circ A)^*\) is \(w^*\)-closed. By (29), we have
\[
\text{cl}^* \Lambda \subseteq \text{epi}(f_1 - f_2 + g_1 \circ A - g_2 \circ A)^*.
\]
Next we will prove that the converse inclusion holds. Let \((p, r) \in \text{epi}(f_1 - f_2 + g_1 \circ A - g_2 \circ A)^*\). By (24), we have that \(v(P_{(A,p)}) \geq -r\), hence \(v(D_{(A,p)}) \geq -r\). Let \(\varepsilon > 0\), there is \(z^* \in Z^*\), for each \((u^*, v^*) \in H^*_2\), satisfying
\[
-(K_1^z + L_1^z \circ A)^*(u^* + p + A^*v^*) + (K_2^z)^*(u^*) + (L_2^z)^*(v^*) \geq -r - \varepsilon.
\]
By Lemma 4.2, \((p, r + \varepsilon) \in \Lambda\). Hence \((p, r) \in \text{cl}^* \Lambda,
\[
\text{epi}(f_1 - f_2 + g_1 \circ A - g_2 \circ A)^* \subseteq \text{cl}^* \Lambda.
\]
This, together with (30), implies that (22) holds. Thus \((ACC)_P\) holds and the proof is complete.

\([=] \) Suppose that \((f_1, f_2, g_1, g_2; A)\) satisfies \((ACC)_P\). Then by Remark 3, \((SCC)_P\) holds. Hence the weak duality holds, i.e., \(v(P_{(A,p)}) \geq v(D_{(A,p)})\), thanks to Theorem 4.3. To show the converse inequality, let \(r = v(P_{(A,p)})\). By \((ACC)_P\) and (24), we have
\[
(p, -r) \in \text{epi}(f_1 - f_2 + g_1 \circ A - g_2 \circ A)^* = \text{cl}^* \Lambda.
\]
There exists a sequence \(\{(p_\alpha, -r_\alpha)\} \subseteq \Lambda\), such that \((p_\alpha, -r_\alpha) \to (p, -r)\). It results from Lemma 4.2 that, \(v(D_{(A,p_\alpha)}) \geq r_\alpha\). Thus we have
\[
\sup_{z^* \in Z^*} \inf_{(u^*, v^*) \in H^*_2} \left\{ -(K_1^z + L_1^z \circ A)^*(u^* + p_\alpha + A^*v^*) + (K_2^z)^*(u^*) + (L_2^z)^*(v^*) \right\} \geq r_\alpha.
\]
By (16), the above inequality can be rewritten as follows,
\[
\sup_{z^* \in Z^*} \inf_{x \in X} \{ K_1^z(x) - K_2^z(x) + L_1^z(Ax) - L_2^z(Ax) - p_\alpha(x) \} \geq r_\alpha.
\]
Since \((p_\alpha, -r_\alpha) \to (p, -r),
\[
\sup_{z^* \in Z^*} \inf_{x \in X} \{ K_1^z(x) - K_2^z(x) + L_1^z(Ax) - L_2^z(Ax) - p(x) \} \geq r,
\]
i.e., \(v(D_{(A,p)}) \geq r\). Thus, \(v(D_{(A,p)}) \geq v(P_{(A,p)})\). This together with the weak duality implies that \(v(P_{(A,p)}) = v(D_{(A,p)})\).

It is noted that, even though the family \((f_1, f_2, g_1, g_2; A)\) satisfies \((ACC)_P\), the stable strong duality does not necessarily hold.

**Example 3.** Let \(X = Y = Z = \mathbb{R}, p = 0\) and \(A\) be the identity operator. Define \(f_1, f_2, g_1, g_2 : \mathbb{R} \to \mathbb{R}\) respectively by \(f_2 = g_i = 0, i = 1, 2,\)
\[
f_1(x) = \begin{cases} 
1, & \text{if } x > 0, \\
-1, & \text{if } x \leq 0.
\end{cases}
\]
Then, \( f_1, f_2, g_1 \) and \( g_2 \) are proper convex functions. Define

\[
F_1(x, z) = \begin{cases} \frac{1}{x} - \sqrt{z}, & \text{if } x > 0, z \geq 0, \\ +\infty, & \text{otherwise}, \end{cases}
\]

\[
F_2(x, z) = G_i(Ax, z) = \begin{cases} 0, & \text{if } z \geq 0, \\ +\infty, & \text{if } z < 0. \end{cases}
\]

Then,

\[
K_1(x, z^*) = \begin{cases} \frac{1}{x} - \frac{1}{4z^*}, & \text{if } x > 0, z^* > 0, \\ -\infty, & \text{if } x > 0, z^* \leq 0, \\ +\infty, & \text{otherwise}, \end{cases}
\]

\[
K_2(x, z^*) = L_i(Ax, z^*) = \begin{cases} 0, & \text{if } z^* \geq 0, \\ -\infty, & \text{if } z^* < 0. \end{cases}
\]

Hence,

\[
\operatorname{epi}(f_1 - f_2 + g_1 \circ A - g_2 \circ A)^* = \{(a, r) : a \leq 0, r \geq -2\sqrt{-a}\},
\]

and

\[
\Lambda = \bigcup_{z^* > 0} \{(a, r) : a \leq 0, r \geq \frac{1}{4z^*} - 2\sqrt{-a}\}.
\]

So, \( \operatorname{epi}(f_1 - f_2 + g_1 \circ A - g_2 \circ A)^* = \text{cl}^* \Lambda \), \( (ACC)_p \) holds and \( v(P_A) = v(D_A) = 0 \). But there is no \( z^* \) where \( v(D_A) \) is attained. Whence the stable strong duality does not hold.

**Theorem 4.4.** The stable strong duality between \( (P_A) \) and \( (D_A) \) holds if and only if the family \( (f_1, f_2, g_1, g_2; A) \) satisfies \( (CC)_p \).

**Proof.** \( [\Rightarrow] \) Suppose the stable strong duality holds, that is, for each \( p \in X^* \), \( v(P(A,p)) = v(D(A,p)) \) and the problem \( (D(A,p)) \) has an optimal solution. There exists \( z^* \in Z^* \) such that

\[
\inf_{(u^*, v^*)} \{- (K_1^* + L_1^* \circ A)^* (u^* + p + A^* v^*) + (K_2^*)^* (u^*) + (L_2^*)^* (v^*)\} = v(D(A,p)).
\]

(31)

By Theorem 4.3, the family \( (f_1, f_2, g_1, g_2; A) \) satisfies \( (SCC)_p \), i.e., (23) holds. We need to prove the converse inclusion, i.e.,

\[
\operatorname{epi}(f_1 - f_2 + g_1 \circ A - g_2 \circ A)^* \subseteq \Lambda.
\]

Let

\[
(p, r) \in \operatorname{epi}(f_1 - f_2 + g_1 \circ A - g_2 \circ A)^*.
\]

Then we have \( v(P(A,p)) \geq -r \), thanks to (24). By the stable strong duality, \( v(D(A,p)) \geq -r \) and there exists \( z^* \in Z^* \) such that (31) holds. Then for each \( (u^*, v^*) \in H_{\Lambda}^* \),

\[
-(K_1^* + L_1^* \circ A)^* (u^* + p + A^* v^*) + (K_2^*)^* (u^*) + (L_2^*)^* (v^*) \geq -r.
\]

By Lemma 4.2, we obtain that \((p, r) \in \Lambda\). This, combined with (32) yields that \( \operatorname{epi}(f_1 - f_2 + g_1 \circ A - g_2 \circ A)^* \subseteq \Lambda \). Whence (21) holds and the family \( (f_1, f_2, g_1, g_2; A) \) satisfies \( (CC)_p \).

\( [\Leftarrow] \) Suppose the family \( (f_1, f_2, g_1, g_2; A) \) satisfies \( (CC)_p \), i.e., (21) holds. By Remark 3, the family \( (f_1, f_2, g_1, g_2; A) \) also satisfies \( (SCC)_p \). Then \( v(P(A,p)) \geq -r \).
Let Corollary 1. If \( v(D(A,p)) = -\infty \), the conclusion trivially holds. To show \( v(D(A,p)) \leq v(D(A,p)) \), let \( r = v(D(A,p)) \in \mathbb{R} \). By (21) and (24),

\[(p, -r) \in \text{epi}(f_1 - f_2 + g_1 \circ A - g_2 \circ A)^* = \Lambda.
\]

By Lemma 4.2, there exists \( z^* \in Z^* \), for each \((u^*, v^*) \in H^*_z\), satisfying

\[-(K_1^{z^*} + L_1^{z^*} \circ A)^*(u^* + p + A^*v^*) + (K_2^{z^*})^*(u^*) + (L_2^{z^*})^*(v^*) \geq r.\]

Thus \( v(D(A,p)) \geq r \). This together with the weak duality implies that \( v(D(A,p)) = v(D(A,p)) = r \). Since \((u^*, v^*) \in H^*_z\) is arbitrary,

\[\inf_{(u^*, v^*) \in H^*_z} \{ -(K_1^{z^*} + L_1^{z^*} \circ A)^*(u^* + p + A^*v^*) + (K_2^{z^*})^*(u^*) + (L_2^{z^*})^*(v^*) \} \geq r = v(D(A,p)).\]

By the definition of problem \( (D(A,p)) \), \( z^* \) is an optimal solution. The stable strong duality holds.

**Corollary 1.** Let \( F_i(x, z) \) be a proper l.s.c. convex function. \( F_i(x, 0) = f_1(x) \) and \( \text{dom} f_i \neq \emptyset \). Then the following statements are equivalent:

(i) \( \inf_{z \in X} \left\{ f_1(x) + x^*(x) \right\} = \max_{z^* \in Z^*} \inf_{z \in X} \{ K_1(x, z^*) + x^*(x) \}, \quad \forall x^* \in X^*.\)

(ii) \( \text{epi} f_1^* = \bigcup_{z^* \in Z^*} \text{epi}(K_1^{z^*})^*.\)

(iii) \( \text{Pr}_{X^* \times \mathbb{R}}(\text{epi} F_1^*) \) is \( w^* \)-closed, where \( K_1 \) and \( K_2 \) are defined in (8) and (13), respectively.

**Proof.** Let \( f_2 = g_i = 0 \) (\( i = 1, 2 \)) and \( A \) be the identity operator. Define

\[F_2(x, z) = G_i(Ax, z) = \begin{cases} 0, & \text{if } z = 0, \\ +\infty, & \text{if } z \neq 0. \end{cases}\]

Then \( K_2(x, z^*) = L_i(Ax, z^*) = 0 \). By Theorem 4.4, (i) is equivalent to (ii). Next we will prove (ii) is equivalent to (iii).

We claim that

\[\bigcup_{z^* \in Z^*} \text{epi}(K_1^{z^*})^* = \text{Pr}_{X^* \times \mathbb{R}}(\text{epi} F_1^*).\]

In fact, \( \text{Pr}_{X^* \times \mathbb{R}}(\text{epi} F_1^*) = \{ (x^*, s) : \exists z_0^* \in Z^*, \text{such that } F_1^*(x^*, z_0^*) \leq s \}. \) Thus

\[(p, r) \in \bigcup_{z^* \in Z^*} \text{epi}(K_1^{z^*})^* \iff \exists \mu \in Z^*, (p, r) \in \text{epi}(K_1^{\mu})^* \]

\[\iff \exists \mu \in Z^*, \sup_{x \in X} \{ p(x) - \inf_{z \in Z} \{ F_1(x, z) + \mu(z) \} \} \leq r \]

\[\iff \exists z_0^* = -\mu \in Z^*, F_1^*(p, z_0^*) \leq r \]

\[\iff (p, r) \in \text{Pr}_{X^* \times \mathbb{R}}(\text{epi} F_1^*).\]

So (33) holds.

We prove that \( \text{Pr}_{X^* \times \mathbb{R}}(\text{epi} F_1^*) \) is convex. Let \((x_1^*, s_1), (x_2^*, s_2) \in \text{Pr}_{X^* \times \mathbb{R}}(\text{epi} F_1^*) \) and \( t \in (0, 1) \). Then there are \( z_1^*, z_2^* \in Z^* \) such that

\[F_1^*(x_1^*, z_1^*) \leq s_1, F_1^*(x_2^*, z_2^*) \leq s_2.\]
Consider $x^* = tx_1^* + (1 - t)x_2^*$, $z^* = tz_1^* + (1 - t)z_2^*$. Then

$$F_1^*(x^*, z^*) = \sup_{x \in X, z \in Z} \{ |tx_1^* + (1 - t)x_2^*|(x) + |tz_1^* + (1 - t)z_2^*|(z) - F_1(x, z) \} \leq t \sup_{x \in X, z \in Z} \{ x_1^*(x) + z_1^*(z) - F_1(x, z) \} + (1 - t) \sup_{x \in X, z \in Z} \{ x_2^*(x) + z_2^*(z) - F_1(x, z) \} = tF_1^*(x_1^*, z_1^*) + (1 - t)F_1^*(x_2^*, z_2^*) \leq ts_1 + (1 - t)s_2.$$ Thus, $(tx_1^* + (1 - t)x_2^*, ts_1 + (1 - t)s_2) \in \text{Pr}_{X \times \mathbb{R}}(\text{epi } F_1^*)$. That is $\text{Pr}_{X \times \mathbb{R}}(\text{epi } F_1^*)$ is convex.

[(ii)⇒(iii)] Suppose (ii) holds, i.e.,

$$\text{epi } f_1^* = \bigcup_{z^* \in Z^*} \text{epi}(K_1^{z*})^*.$$ Clearly, $f_1^*$ is l.s.c. and convex thanks to [20, Theorem 2.3.1]. Then $\text{epi } f_1^*$ is $w^*$-closed and convex thanks to [20, Theorem 2.2.1]. By (33), we have

$$\text{epi } f_1^* = \text{Pr}_{X \times \mathbb{R}}(\text{epi } F_1^*).$$ Whence (iii) holds.

[(iii)⇒(ii)] Suppose that (iii) holds. Since $\text{Pr}_{X \times \mathbb{R}}(\text{epi } F_1^*)$ is convex, $\text{Pr}_{X \times \mathbb{R}}(\text{epi } F_1^*)$ is $w^*$-closed and convex. By (33), $\bigcup_{z^* \in Z^*} \text{epi}(K_1^{z*})^*$ is also $w^*$-closed and convex. It follows from (1) and Lemma 3.2 that,

$$\text{epi}\left( \sup_{z^* \in Z^*} (\text{cl } K_1^{z*}) \right)^* = \text{cl}^\ast \text{co}\left( \bigcup_{z^* \in Z^*} \text{epi}(\text{cl } K_1^{z*})^* \right) = \text{cl}^\ast \text{co}\left( \bigcup_{z^* \in Z^*} \text{epi}(K_1^{z*})^* \right) = \bigcup_{z^* \in Z^*} \text{epi}(K_1^{z*})^*.$$ Since $\text{cl } K_1^{z*} \leq K_1^{z*}$,

$$\left( \sup_{z^* \in Z^*} (\text{cl } K_1^{z*}) \right)^* \supseteq \left( \sup_{z^* \in Z^*} K_1^{z*} \right)^*.$$ To prove the converse inequality, let $x^* \in X^*$ and $\alpha \in \mathbb{R}$ be such that

$$\left( \sup_{z^* \in Z^*} K_1^{z*} \right)^* (x^*) \leq \alpha.$$ Then $x^* - \sup_{z^* \in Z^*} K_1(x, z^*) \leq \alpha$ for every $x \in X$. Denote $\phi(x) := x^*(x) - \alpha$. Then

$$\phi(x) \leq \sup_{z^* \in Z^*} K_1(x, z^*).$$ Let $\varepsilon > 0$. Then there exists $z^* \in Z^*$ such that $\phi(x) - \varepsilon \leq K_1(x, z^*)$. So $\text{epi}(\phi - \varepsilon) \supseteq \text{epi } K_1^{z*}$. Since $\phi(x) - \varepsilon$ is a proper l.s.c. convex function, we have

$$\text{epi}(\phi - \varepsilon) \supseteq \text{cl} \text{epi } (K_1^{z*}) = \text{epi}(\text{cl } K_1^{z*}).$$
Therefore, \( \varphi(x) - \varepsilon \leq \text{cl } K_1^z(x) \) for every \( x \in X \). Then \( \varphi(x) \leq \sup_{z^* \in Z^*} (\text{cl } K_1^z(x)) \) and for every \( x \)

\[
x^*(x) - \sup_{z^* \in Z^*} (\text{cl } K_1^z(x)) \leq \alpha,
\]

which implies that \( \left( \sup_{z^* \in Z^*} (\text{cl } K_1^z(x)) \right)^*(x^*) \leq \alpha \). We have

\[
\left( \sup_{z^* \in Z^*} (\text{cl } K_1^z(x)) \right)^* \leq \left( \sup_{z^* \in Z^*} K_1^z \right)^*.
\]

This together with (36) implies \( \left( \sup_{z^* \in Z^*} (\text{cl } K_1^z(x)) \right)^* = \left( \sup_{z^* \in Z^*} K_1^z \right)^* \). By (35), we have

\[
\text{epi}\left( \sup_{z^* \in Z^*} K_1^z \right)^* = \bigcup_{z^* \in Z^*} \text{epi}(K_1^z)^*.
\]

By (12), \( \text{epi} f_1^* = \text{epi}\left( \sup_{z^* \in Z^*} K_1^z \right)^* \). Thus,

\[
\text{epi} f_1^* = \bigcup_{z^* \in Z^*} \text{epi}(K_1^z)^*.
\]

(ii) holds and the proof is complete. \( \square \)

**Remark 4.** By the proof of (34), (i) of Corollary 1 can be rewritten as follows

\[
\inf_{x \in X} \{ F_1(x,0) + x^*(x) \} = \max_{z^* \in Z^*} \{-F_1^*(-x^*,z^*)\}.
\]

So the equivalence of (i) and (iii) was also given in [7, Theorem 3.1], which is called the stable conjugate duality.

**Corollary 2.** Suppose that the family \( (f_1, f_2, g_1, g_2; A) \) satisfies (CC)p. For each \( p \in X^* \) and each \( \alpha \in \mathbb{R} \), the following statements are equivalent:

(i) \( x \in X \Rightarrow f_1(x) - f_2(x) + g_1(Ax) - g_2(Ax) - p(x) \geq \alpha \).

(ii) there exists \( z^* \in Z^* \), for each \( (u^*, v^*) \in H_p^* \), such that

\[
-(K_1^z + L_1^z \circ A)^*(u^* + p + A^*v^*) + (K_2^z)^*(u^*) + (L_2^z)^*(v^*) \geq \alpha.
\]

5. **Stable total dualities.** Recall that \( f_i, g_i \ (i = 1, 2) \) are proper convex functions, \( A \) is a linear continuous operator, \( F_i, G_i \) are proper convex functions, such that \( F_i(x,0) = f_i(x), G_i(Ax,0) = g_i(Ax), F_i(x,\cdot), G_i(y,\cdot) \) are closed. In order to study the total dualities, for each \( p \in X^* \), we use \( S_p(p) \) to denote the optimal solution set of \( (P_{(A,p)}) \).

Below we will make use of the subdifferential \( \partial h(x) \) (see (3)) for a general proper function (not necessarily convex) \( h : X \to \mathbb{R} \). Clearly, the following equivalence holds:

\[
x_0 \text{ is a minimizer of } h \text{ if and only if } 0 \in \partial h(x_0).
\]

In the following definition, following [20, page 2], we adapt the convention that \( \cap_{t \in S} t = \emptyset \).

**Definition 5.1.** Let \( x_0 \in X, z_0^* \in Z^* \). The family \( (f_1, f_2, g_1, g_2; A) \) is said to satisfy the condition \( (GBCQ) \) at \( (x_0, z_0^*) \) if

\[
\partial (f_1 - f_2 + g_1 \circ A - g_2 \circ A)(x_0) = \bigcap_{(u^*, v^*) \in H_p^*} \partial (K_1^z + L_1^z \circ A)(x_0) - u^* - A^*v^* \tag{38}
\]
where $H_{x_0}^*$, is defined by (17). We say that the family of $(f_1, f_2, g_1, g_2; A)$ satisfies the condition $(GBCQ)$, if for each $x_0 \in D$, there exists $z_0^* \in Z^*$ such that $(GBCQ)$ is valid at $(x_0, z_0^*)$.

**Example 4.** Let $X = Y = Z = \mathbb{R}$ and $A$ be the identity operator. Define $f_1, f_2, g_1, g_2 : \mathbb{R} \to \mathbb{R}$ respectively by $f_2 = g_i = 0$, $i = 1, 2$,

$$f_1(x) = \begin{cases} 0, & \text{if } x > 1, \\ 1, & \text{if } x = 1, \\ +\infty, & \text{if } x < 1. \end{cases}$$

Then, $f_1, f_2, g_1$ and $g_2$ are proper convex functions. Define

$$F_1(x, z) = f_1(x) + z, \quad G_i(Ax, z) = g_i(x) + z.$$

Then,

$$K_1(x, z^*) = \begin{cases} f_1(x), & \text{if } z^* = -1, \\ -\infty, & \text{if } z^* \neq -1, \end{cases}$$

$$K_2(x, z^*) = L_i(Ax, z^*) = \begin{cases} 0, & \text{if } z^* = -1, \\ -\infty, & \text{if } z^* \neq -1. \end{cases}$$

Take $z^* = -1$, we have

$$(K_2^{-1})^*(x^*) = (L_2^{-1})^*(x^*) = \begin{cases} 0, & \text{if } x^* = 0, \\ +\infty, & \text{if } x^* \neq 0. \end{cases}$$

So $dom(K_2^{-1})^* = dom(L_2^{-1})^* = \{0\}$. Since

$$\partial(f_1 - f_2 + g_1 \circ A - g_2 \circ A)(x) = \partial f_1(x) = \begin{cases} \{0\}, & \text{if } x > 1, \\ \emptyset, & \text{if } x \leq 1. \end{cases}$$

whence, (38) holds, $(f_1, f_2, g_1, g_2; A)$ satisfies the condition $(GBCQ)$.

**Lemma 5.2.** Let $x_0 \in X$ and $p \in X^*$ be such that $x_0 \in S_P(p)$. Assume that there exists $z^* \in Z^*$ such that

$$f_1(x_0) + g_1(Ax_0) = K_1(x_0, z^*) + L_1(Ax_0, z^*), \quad \text{(39)}$$

and

$$p \in \bigcap_{(u^*, v^*) \in H_2^*} \{ \partial(K_1^* + L_1^* \circ A)(x_0) - u^* - A^*v^* \}. \quad \text{(40)}$$

Then $v(D_{(A, p)}) \geq v(P_{(A, p)})$ and each $(u^*, v^*) \in H_2^*$ satisfies

$$-(K_1^* + L_1^* \circ A)(u^* + p + A^*v^*) + (K_2^*)(u^*) + (L_2^*)(v^*) \geq v(P_{(A, p)}). \quad \text{(41)}$$

**Proof.** Take $(u^*, v^*) \in \partial H_2^*(x_0)$, such that

$$p \in \partial(K_1^* + L_1^* \circ A)(x_0) - u^* - A^*v^*.$$  

Consequently, there exists $x^* \in \partial(K_1^* + L_1^* \circ A)(x_0)$ such that

$$p = x^* - u^* - A^*v^*. \quad \text{(42)}$$

By the equality in (4), we have

$$x^*(x_0) = (K_1^* + L_1^* \circ A)^*(x^*) + K_1^*(x_0) + L_1^*(Ax_0).$$

The above formula is rewritten as follows,

$$(K_1^* + L_1^* \circ A)(x^*) = x^*(x_0) - K_1(x_0, z^*) - L_1(Ax_0, z^*). \quad \text{(43)}$$
By the Young-Fenchel inequality (4), we have
\[ u^*(x_0) \leq (K_2^* + L_1^* \circ A)^*(u^* + p + A^*v^*) + (K_2^* + L_2^*)^*(v^*) + L_2^*(Ax_0). \]

So
\[ (K_2^* + L_1^* \circ A)^*(u^*) \geq u^*(x_0) - K_2(x_0, z^*), (K_2^* + L_2^*)^*(v^*) \geq v^*(Ax_0) - L_2(Ax_0, z^*). \]

Thus
\[
-K_1^* + L_1^* \circ A)^*(u^* + p + A^*v^*) + (K_2^* + L_2^*)^*(v^*) \geq K_1(x_0, z^*) + L_1(Ax_0, z^*) - K_2(x_0, z^*) - L_2(Ax_0, z^*) - x^*(x_0) + u^*(x_0) + v^*(Ax_0) = K_1(x_0, z^*) + L_1(Ax_0, z^*) - K_2(x_0, z^*) - L_2(Ax_0, z^*) - p(x_0) \geq f_1(x_0) + g_1(Ax_0) - f_2(x_0) - g_2(Ax_0) - p(x_0),
\]

where the first and the second equalities are due to (42), the first inequality is due to (43) and (44), and the last inequality is due to (12) and (39).

Since \( x_0 \in S_p(p) \),
\[-(K_1^* + L_1^* \circ A)^*(u^* + p + A^*v^*) + (K_2^* + L_2^*)^*(v^*) \geq v^*(P_{(A,p)}) \]
which implies that (41) holds. By the definition of \( D_{(A,p)} \), \( v(D_{(A,p)}) \geq v(P_{(A,p)}) \).

The proof is complete. \( \square \)

**Theorem 5.3.** Suppose that the family \((f_1, f_2, g_1, g_2; A)\) satisfies \((SCC)_p\). For each \( p \in X^* \), there exist \( z^*_0 \in Z^* \), \( x_0 \in S_p(p) \), such that \( K_2^0 \) and \( L_2^0 \) are proper l.s.c functions, \( f_1(x_0) + g_1(Ax_0) = K_1(x_0, z^*_0) + L_1(Ax_0, z^*_0) \) and the family \((f_1, f_2, g_1, g_2; A)\) satisfies the condition \((GBCQ)\) at \((x_0, z^*_0)\). Then the stable total duality holds.

**Proof.** By (37), \( p \in \partial(f_1 - f_2 + g_1 \circ A - g_2 \circ A)(x_0) \). Hence, (40) holds by the assumed condition \((GBCQ)\) at \((x_0, z^*_0)\). Now Lemma 5.2 is applied to get that \( v(D_{(A,p)}) \geq v(P_{(A,p)}) \) and (41) holds. Moreover, \((SCC)_p\) holds, \( v(D_{(A,p)}) \leq v(P_{(A,p)}) \) thanks to Theorem 4.3. So \( v(D_{(A,p)}) = v(P_{(A,p)}) \). By (41), we have
\[
\inf_{(u^*, v^*) \in H^*_{z^*_0}} \{ -(K_1^0 + L_1^0 \circ A)^*(u^* + p + A^*v^*) + (K_2^0)^*(u^*) + (L_2^0)^*(v^*) \} \geq v(D_{(A,p)}),
\]

which combined with the definition of \( D_{(A,p)} \) implies that \( z^*_0 \) is the optimal solution. Thus, the stable total duality holds and the proof is complete. \( \square \)

In the case when \( f_2 = g_2 = 0 \), we have the following corollary.

**Corollary 3.** Suppose that for each \( p \in X^* \), there exist \( z^*_0 \in Z^* \), \( x_0 \in S_p(p) \), such that \( f_1(x_0) + g_1(Ax_0) = K_1(x_0, z^*_0) + L_1(Ax_0, z^*_0) \) and the family \((f_1, g_1; A)\) satisfies the \((GBCQ)\) at \((x_0, z^*_0)\). Then the stable total duality holds.

**Acknowledgments.** The authors would like to thank the reviewers for their valuable comments and suggestions which improved the presentation of the paper.
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Received June 2011; 1st revision October 2012; 2nd revision March 2013.

E-mail address: ligang@hitsz.edu.cn
E-mail address: mayangxq@polyu.edu.hk
E-mail address: yuyingz@suda.edu.cn