STABILITY OF BOUNDARY LAYERS FOR A VISCOUS HYPERBOLIC SYSTEM ARISING FROM CHEMOTAXIS: ONE DIMENSIONAL CASE

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Abstract. This paper is concerned with the stability of boundary layer solutions for a viscous hyperbolic system transformed via a Cole-Hopf transformation from a singular chemotactic system modeling the initiation of tumor angiogenesis proposed in [35]. It was previously shown in [26] that when prescribed with Dirichlet boundary conditions, the system possesses boundary layers at the boundaries in an bounded interval (0, 1) as the chemical diffusion rate (denoted by \( \varepsilon > 0 \)) is small. This paper proceeds to prove the stability of boundary layer solutions and identify the precise structure of boundary layer solutions. Roughly speaking, we justify that the solution with \( \varepsilon > 0 \) converges to the solution with \( \varepsilon = 0 \) (outer layer solution ) plus the inner layer solution with the optimal rate at order of \( O(\varepsilon^{1/2}) \) as \( \varepsilon \to 0 \), where the outer and inner layer solutions are well determined and relation between outer and inner layer solutions can be explicitly identified. Finally we transfer the results to the original pre-transformed chemotaxis system and discuss the implications of our results.

1. Introduction

Chemotaxis, the movement of an organism in response to a chemical stimulus, has been an important mechanism of various biological phenomena/processes, such as aggregation of bacteria [49, 63], slime mould formation [23], fish pigmentation [53], tumor angiogenesis [7–9], primitive streak formation [54], blood vessel formation [17], wound healing [56]. The prototypical chemotaxis model, known as Keller-Segel (KS) model due to their pioneering works of [30–32], reads in its general form as

\[
\begin{align*}
    u_t &= [Du_{x} - \chi u(\phi(c))_x]_x, \\
    c_t &= \varepsilon c_{xx} + g(u, c),
\end{align*}
\]  

(1.1)

where \( u(x, t) \) and \( c(x, t) \) denote the cell density and chemical (signal) concentration at position \( x \) and time \( t \), respectively. The function \( \phi(c) \) is called the chemotactic sensitivity function accounting for the signal response mechanism and \( g(u, c) \) is the chemical kinetics (birth and death). \( D > 0 \) and \( \varepsilon \geq 0 \) are cell and chemical diffusion coefficients, respectively. \( \chi \neq 0 \) is referred to as the chemotactic coefficient with \( |\chi| \) measuring the strength of the chemotactic sensitivity, where the chemotaxis is said to be attractive if \( \chi > 0 \) and repulsive if \( \chi < 0 \). The application of (1.1) generically depends on the specific forms of \( \phi(c) \) and \( g(u, c) \). There are two major classes of chemotactic sensitivity functions: linear law \( \phi(c) = c \) and logarithmic law \( \phi(c) = \ln c \). The former was originally derived in [30, 31] by Keller and Segel to model the self-aggregation of \textit{Dictyostelium discoideum} in response to cyclic adenosine monophosphate (cAMP), while the latter was first employed in [32] to model the wave propagation of bacterial chemotaxis though it has many other prominent applications in biology (cf. [2, 3, 10, 29, 34, 52]). Compared with massive well-known results on the KS system with linear chemotactic sensitivity (cf. [4, 5, 22, 25]), not much results are available for the logarithmic sensitivity due to its singularity nature (at \( c = 0 \)). This paper is concerned with the following KS system with logarithmic sensitivity:

\[
\begin{align*}
    u_t &= [Du_{x} - \chi u(\ln c)_x]_x, \\
    c_t &= \varepsilon c_{xx} - \mu uc,
\end{align*}
\]  

(1.2)
which was a specialized KS model with linear nutrient consumption proposed in [30], and later found applications in [35] to describe the dynamical interactions between vascular endothelial cells (VECs), denoted by \( u \), and signaling molecules vascular endothelial growth factor (VEGF), denoted by \( c \), in the initiation of tumor angiogenesis. Except this, the model (1.2) has also been used in [50] to model the boundary movement of chemotactic bacterial populations. Though bearing specific applications, the logarithmic sensitivity brings considerable challenges to mathematical analysis due to its singularity nature. The common approach currently used to overcome this singularity is the following Cole-Hopf type transformation (cf. [34, 43]):

\[
v = -\frac{\sqrt{\chi \mu}}{\mu} (\ln c)_x = -\frac{\sqrt{\chi \mu}}{\mu} c_x,
\]

which transforms the model (1.2) into a non-singular system of conservation laws

\[
\begin{cases}
  u_t - (uv)_x = u_{xx}, \\
  v_t - (u - \frac{\epsilon}{\chi} v^2)_x = \frac{\epsilon}{D} v_{xx}, \\
  (u, v)(x, 0) = (u_0, v_0)(x),
\end{cases}
\]

(1.4)

where the temporal-spatial rescalings \( \tilde{t} = \frac{\mu}{\chi} t \) and \( \tilde{x} = \frac{\sqrt{\mu}}{\chi} x \) have been used and tildes have been suppressed in (1.4) for convenience.

Though the transformed system (1.4) no longer has singularity, it has a quadratic nonlinear convection term and the parameter \( \epsilon \) in (1.4) plays a dual role: coefficient of both diffusion and nonlinear convection, which is a prominent feature compared to existing viscous hyperbolic systems as far as we know (cf. [16, 19, 60]). How to make a balance between the diffusion and nonlinear convection becomes an art of analysis. The transformed model (1.4) in multi-dimensions still remains poorly understood so far and available results are limited to small-data solutions (cf. [11, 21, 36, 40, 55, 66]). In contrast the model (1.4) has been well understood in one dimension to a large extent such as the existence and stability of traveling wave solutions (cf. [6, 28, 39, 41–44]) and large-data solutions (cf. [37, 48]) in \( \mathbb{R} \) or in bounded intervals with various boundary conditions (cf. [40, 61, 68, 73]).

The present paper will be to investigate the zero-limit problem of (1.4) as \( \epsilon \to 0 \), which is motivated by the fact pointed out in [35] that the magnitude of the diffusion rate \( \epsilon \) of the chemical VEGF can be negligible compared to the diffusion of VECs in the initiation of tumor angiogenesis. Moreover the diffusion rate \( \epsilon \) was assumed to be zero in the analysis of [32] and many subsequent works (cf. [65]) for simplicity. Hence whether the non-diffusive model (i.e. \( \epsilon = 0 \)) is a good approximation of the diffusive model when \( \epsilon > 0 \) is small is of importance. This promises a relevance to explore the zero-limit problem of (1.4) in elucidating this question. From mathematical point of view, the zero-limit problem of (1.4) as \( \epsilon \to 0 \) is of independent interest due to the dual role of \( \epsilon \) which causes challenges in deriving uniform-in-\( \epsilon \) estimates. This topic has been investigated in several circumstances. First in unbounded domains, it has been shown that both traveling wave solutions (cf. [65]) in \( \mathbb{R} \) and the global small-data solution of the Cauchy problem (cf. [55, 66]) in \( \mathbb{R}^N(N = 2, 3) \) are uniformly convergent in \( \epsilon \), namely the solutions of (1.4) with \( \epsilon > 0 \) converge to those with \( \epsilon = 0 \) in \( L^\infty \)-norm as \( \epsilon \to 0 \). However the \( \epsilon \)-limit problem in bounded domains appears to be more involved. This is closely related to the boundary layer theory, which has been an important topic in fluid mechanics stimulating a large body of outstanding works (cf. [13, 14, 16, 19, 20, 27, 60, 64, 70, 72]). A fundamental question in fluid mechanics is whether solutions of the incompressible Navier-Stokes equations (NSE) converge to those of the Euler equations as the viscosity vanishes. The positive answer of this question has been given to the incompressible NSE under Lions- or Navier-type boundary conditions (cf. [45, 69]). However the convergence under no-slip (zero Dirichlet) boundary condition is elusive due to the appearance of degenerate Prandtl-type boundary layers (cf. [1, 18, 46, 51, 71]). This imposes an interesting question: under what type of boundary conditions, the solutions of (1.4) converge as \( \epsilon \to 0? \) This topic has been recently studied in [38, 67] in a bounded interval. Hereafter we assume \( D = \chi = \mu = 1 \) without loss of generality for simplicity since the specific values of them are not of importance to our analysis. For illustration, let’s first consider the
initial-boundary value problem (IBVP) of system (1.4) in an interval (0, 1):

\[
\begin{align*}
&u_t - (uv)_x = u_{xx}, \quad (x, t) \in (0, 1) \times (0, \infty), \\
v_t - (u - \varepsilon |v|^2)_x = \varepsilon v_{xx}, \\
&(u, v)(x, 0) = (u_0, v_0)(x).
\end{align*}
\]  

(1.5)

If (1.5) is endowed with the mixed homogeneous Neumann-Dirichlet boundary conditions

\[ u_{x|x=0,1} = v_{x|x=0,1} = 0, \]

it was shown in [67] that the solutions of (1.5) are uniformly convergent in \( \varepsilon \), namely the solutions of diffusive problem \( (\varepsilon > 0) \) uniformly converge to those of non-diffusive problem \( (\varepsilon = 0) \). However if the Dirichlet boundary conditions are imposed, one cannot impose Dirichlet boundary conditions for \( v \) at boundaries if \( \varepsilon = 0 \) since otherwise the non-diffusive problem \( (\varepsilon = 0) \) may be over-determined (cf. [38]). This indicates that the Dirichlet boundary conditions of (1.5) ought to be prescribed as:

\[
\begin{align*}
&u|_{x=0,1} = \bar{u} \geq 0, \quad v|_{x=0,1} = \bar{v}, \quad \text{if } \varepsilon > 0, \\
&u|_{x=0,1} = \bar{u} \geq 0, \quad \text{if } \varepsilon = 0
\end{align*}
\]  

(1.6)

where the boundary values of \( v \) when \( \varepsilon = 0 \) are determined by the second equation of (1.5) via \( u_x \), and may not equal to \( \bar{v} \). Due to this possible mismatch of boundary conditions between \( \varepsilon > 0 \) and \( \varepsilon = 0 \), the \( L^\infty \)-norm of \( v \) may diverge as \( \varepsilon \to 0 \) near end points \( x = 0, 1 \), and if so boundary layers will arise. Such suspicion has been numerically verified recently by Li and Zhao in [38], followed with a rigorous proof in [26]. Precisely speaking, if letting \( (u^\varepsilon, v^\varepsilon) \) and \( (u^0, v^0) \) denote the solutions of the IBVP (1.5)-(1.6) for \( \varepsilon > 0 \) and \( \varepsilon = 0 \), respectively, then the work [26] showed that if initial data satisfy some compatibility conditions on boundaries, then for any function \( \delta(\varepsilon) \) depending on \( \varepsilon \) and satisfying \( \delta(\varepsilon) \to 0 \) and \( \varepsilon^{1/2} / \delta(\varepsilon) \to 0 \) as \( \varepsilon \to 0 \), it holds that for any \( T > 0 \),

\[
\begin{align*}
&\lim_{\varepsilon \to 0} \| u^\varepsilon - u^0 \|_{L^\infty([0,1] \times [0,T])} = 0, \\
&\lim_{\varepsilon \to 0} \| v^\varepsilon - v^0 \|_{L^\infty([\delta,1-\delta] \times [0,T])} = 0, \quad \liminf_{\varepsilon \to 0} \| v^\varepsilon - v^0 \|_{L^\infty([0,1] \times [0,T])} > 0,
\end{align*}
\]  

(1.7)

where the function \( \delta(\varepsilon) \) is called a boundary layer (BL) thickness following the nomenclature of [14, 15]. But, it is easy to see that the BL-thickness \( \delta(\varepsilon) \) satisfying the above constraints is not unique, for example the relations given in (1.7) hold for any \( \delta(\varepsilon) = \varepsilon^\alpha \) with \( \alpha < \frac{1}{2} \). A formal asymptotic analysis was further performed in [26] to show that the exact BL-thickness magnitude is of order \( \varepsilon^{1/2} \).

Compared to the boundary layer theory developed in fluid mechanics, the study of boundary layer theory of chemotaxis models is still in its infant stage. The rigorous work [26] only showed the existence of boundary layer solutions of (1.5)-(1.6) and proved the convergence of the solution component \( v^\varepsilon \) as \( \varepsilon \to 0 \) outside the boundary layers. However, the structure of \( v^\varepsilon \) as \( \varepsilon \to 0 \) inside the boundary layers remains open. In this paper we shall exploit the structure of \( v^\varepsilon \) inside the boundary layers and hence establish the stability of boundary layer solutions of (1.5)-(1.6) in the entire interval \( (0,1) \). With the general boundary layer theory [57, 59] applied to (1.5)-(1.6), the solution profile \( (u^\varepsilon, v^\varepsilon) \) of (1.5) for small \( \varepsilon > 0 \) is composed of two parts: outer layer profile and inner (boundary) layer profile. Since \( u^\varepsilon \) converges uniformly in \( \varepsilon \) and hence the inner layer profile part will be absent, \( (u^\varepsilon, v^\varepsilon) \) is anticipated to possess the form:

\[
\begin{align*}
&u^\varepsilon = u^0 + O(\varepsilon^\alpha); \\
v^\varepsilon = v^0 + v^L \left( \frac{x}{\sqrt{\varepsilon}}, t \right) + v^R \left( \frac{x-1}{\sqrt{\varepsilon}}, t \right) + O(\varepsilon^\alpha)
\end{align*}
\]  

(1.8)

for some \( \alpha \leq 1/2 \), where \( (u^0, v^0) \) is the outer layer profile which is the solution of non-diffusive problem of (1.5)-(1.6) with \( \varepsilon = 0 \), and the inner (boundary) layer profile \( v^L/v^R \) adjust rapidly from a value away from the boundary to a different value on the left/right end point. Outside the boundary layer, the non-diffusive problem dominates. Inside the boundary layer, diffusion becomes important. The main goal of this paper is to explicitly derive the outer/inner layer profiles and justify (1.8) holds as \( \varepsilon \to 0 \) for \( \alpha = 1/2 \), which is the optimal convergence rate since...
the magnitude of boundary layer thickness is of order $\varepsilon^{1/2}$. Finally we convert the results of (1.5)-(1.6) back to the original chemotaxis model (1.2) and find that the chemical concentration has no boundary layer but its gradient does. This essentially means that chemotactic flux near the boundary will change drastically if the chemical diffusion rate is small, which implies that the chemical diffusion plays an important role in the tumor angiogenesis (see more detailed discussion for the biological implications of our results in the end of section 2).

2. Statement of Main results

Notations. For clarity, we specify some notations below.

- Without loss of generality, we assume $0 \leq \varepsilon < 1$ throughout this paper for we consider the diffusion limit problem as $\varepsilon \to 0$;
- Unless specified, we use $C$ to denote a generic positive constant which is independent of $\varepsilon$, depends on time variable and may vary in the context, while $C_0$ denotes a generic positive constant independent of $\varepsilon$ and time $t$;
- $L^p$ with $1 \leq p \leq \infty$ represents the Lebesgue space $L^p(0,1)$ with respect to $x \in (0,1)$. $L^p_\varepsilon$ and $L^p_\xi$ denote $L^p(0,\infty)$ with respect to $z$ and $L^p(-\infty,0)$ with respect to $\xi$, respectively. Similarly, $H^k, H^k_z$ and $H^k_\xi$ denote the Sobolev spaces $W^{k,2}$ in $(0,1), (0,\infty)$ and $(-\infty,0)$ with respect to $x, z$ and $\xi$, respectively;
- Denote $(z) = \sqrt{1+z^2}$ for $z \in [0, \infty)$, and $(\xi) = \sqrt{1+\xi^2}$ for $\xi \in (\infty,0]$;
- $\mathbb{N}$ denotes the set of nonnegative integers, and $\mathbb{N}_+$ represents the set of positive integers.

2.1. Boundary layer profiles. In this subsection, we are devoted to using formal asymptotic analysis to find the equations of boundary layer profiles of (1.5) with small $\varepsilon > 0$. The boundary layer thickness has been formally justified as $\mathcal{O}(\varepsilon^{1/2})$ in appendix of [26]. Thus based on the WKB method (cf. [19, 24, 58]), solutions of (1.5) with $\varepsilon > 0$ have the following expansions for $j \in \mathbb{N}$:

\[
\begin{align*}
    u_\varepsilon(x,t) &= \sum_{j=0}^{\infty} \varepsilon^{j/2} \left( u^{I,j}(x,t) + u^{B,j}(z,t) + u^{b,j}(\xi,t) \right), \\
v_\varepsilon(x,t) &= \sum_{j=0}^{\infty} \varepsilon^{j/2} \left( v^{I,j}(x,t) + v^{B,j}(z,t) + v^{b,j}(\xi,t) \right),
\end{align*}
\]

(2.1)

with boundary layer coordinates (or stretching transformations) defined as:

\[
z = \frac{x}{\sqrt{\varepsilon}}, \quad \xi = \frac{x - 1}{\sqrt{\varepsilon}}, \quad x \in [0,1],
\]

(2.2)

where each term in (2.1) is assumed to be smooth, and the boundary layer profiles $(u^{B,j}, v^{B,j})$ and $(u^{b,j}, v^{b,j})$ enjoy the following basic hypothesis (cf. [24, Chapter 4], [19], [58]):

(H) $u^{B,j}$ and $v^{B,j}$ decay to zero exponentially as $z \to \infty$, while $u^{b,j}$ and $v^{b,j}$ decay to zero exponentially as $\xi \to -\infty$ for all $j \geq 0$.

To derive the equations of boundary layer profiles in (2.1), we split our analysis into three steps. We first insert expansions (2.1) into the initial data in (1.5) and into (1.6) to obtain the initial and boundary values of outer and inner layer profiles. Then in the second and third steps, equations for both outer and inner layer solutions will be derived by substituting (2.1) into the first and second equations of (1.5) successively. Proceeding with these procedures by the asymptotic matching method (details are given in Appendix), we derive that the leading-order outer layer solution pair $(u^{I,0}, v^{I,0})(x,t)$ satisfies the following problem:

\[
\begin{align*}
u^{I,0}_t &= (u^{I,0}, v^{I,0})_x + u^{I,0}_{xx}, & (x,t) \in (0,1) \times (0,\infty), \\
v^{I,0} &= u^{I,0}, \\
(u^{I,0}, v^{I,0})(x,0) &= (u_0, v_0)(x), \\
u^{I,0}(0,t) &= u^{I,0}(1,t) = \bar{u},
\end{align*}
\]

(2.3)
which is exactly the non-diffusive form of (1.5)-(1.6) with \( \varepsilon = 0 \). Thus \((u^0, v^0)\) solves (2.3) by uniqueness. The leading-order inner layer solution \( v^{B,0}(z, t) \) near the left end point of \((0, 1)\) satisfies
\[
\begin{cases}
    v^{B,0}_z = -\bar{u} v^{B,0} + v^{B,0}_{zz}, & (z, t) \in (0, \infty) \times (0, \infty), \\
v^{B,0}(z, 0) = 0, \\
v^{B,0}(0, t) = \bar{v} - v^{I,0}(0, t),
\end{cases}
\]
(2.4)
and \( u^{B,0}(z, t) \equiv 0 \), and the first-order inner layer solution \( u^{B,1}(z, t) \) is determined by \( v^{B,0}(z, t) \) through
\[
u^{B,1}(z, t) = \bar{u} \int_{z}^{\infty} v^{B,0}(y, t) \, dy, \quad z \in [0, \infty).
\]
(2.5)
The leading-order inner layer solution \( v^{b,0}(\xi, t) \) near the right end point of \((0, 1)\) satisfies
\[
\begin{cases}
    v^{b,0}_t = -\bar{u} v^{b,0} + v^{b,0}_{\xi \xi}, & (\xi, t) \in (-\infty, 0) \times (0, \infty), \\
v^{b,0}(\xi, 0) = 0, \\
v^{b,0}(0, t) = \bar{v} - v^{I,0}(1, t),
\end{cases}
\]
(2.6)
and \( u^{b,0}(\xi, t) \equiv 0 \), and the corresponding first-order inner layer solution \( u^{b,1}(\xi, t) \) is given by
\[
u^{b,1}(\xi, t) = \bar{u} \int_{\xi}^{-\infty} v^{b,0}(y, t) \, dy, \quad \xi \in (-\infty, 0].
\]
(2.7)

To carry out our desired results, we need the estimates of the first-order outer layer solution pair \((u^{I,1}, v^{I,1})(x, t)\) which satisfies the following problem:
\[
\begin{cases}
    u^{I,1}_x = (u^{I,0} v^{I,1})_x + (u^{I,1} v^{I,0})_x + u^{I,1}_x u^{I,0}, & (x, t) \in (0, 1) \times (0, \infty), \\
v^{I,1}_t = u^{I,1}_x, \\
(u^{I,1}, v^{I,1})(x, 0) = (0, 0), \\
u^{I,1}(0, t) = -\bar{u} \int_{0}^{\infty} v^{I,0}(z, t) \, dz, \\
u^{I,1}(1, t) = -\bar{u} \int_{0}^{-\infty} v^{b,0}(\xi, t) \, d\xi.
\end{cases}
\]
(2.8)
Moreover the inner layer profile \((u^{B,2}, v^{B,1})(z, t)\) satisfies
\[
\begin{cases}
    v^{B,1}_z = -\bar{u} v^{B,1} + v^{B,1}_{zz} - 2(v^{I,0}(0, t) + v^{B,0}) v^{B,0}_z + \int_{z}^{\infty} \Phi(y, t) \, dy, \\
v^{B,1}(z, 0) = 0, \\
v^{B,1}(0, t) = -u^{I,1}(0, t),
\end{cases}
\]
(2.9)
and
\[
u^{B,2}(z, t) = \bar{u} \int_{z}^{\infty} v^{B,1}(y, t) \, dy - \int_{z}^{\infty} \int_{y}^{\infty} \Phi(s, t) \, ds \, dy,
\]
(2.10)
where \( \Phi(z, t) := (u^{I,1}(0, t) + u^{B,0}) v^{I,0}_z + u^{I,0}_z(0, t) v^{B,0} + u^{B,1}_z(v^{I,0}(0, t) + v^{B,0}) + z u^{I,0}_x(0, t) v^{B,0}_z \).

Correspondingly the inner layer profile \((u^{b,2}, v^{b,1})(\xi, t)\) satisfies
\[
\begin{cases}
    v^{b,1}_t = -\bar{u} v^{b,1} + v^{b,1}_{\xi \xi} - 2(v^{I,0}(1, t) + v^{b,0}) v^{b,0}_\xi + \int_{\xi}^{\infty} \Psi(y, t) \, dy, \\
v^{b,1}(\xi, 0) = 0, \\
v^{b,1}(0, t) = -u^{I,1}(1, t),
\end{cases}
\]
(2.11)
and
\[
u^{b,2}(\xi, t) = \bar{u} \int_{\xi}^{-\infty} v^{b,1}(y, t) \, dy - \int_{\xi}^{-\infty} \int_{y}^{-\infty} \Psi(s, t) \, ds \, dy,
\]
(2.12)
In [38], the authors proved the

One can derive the initial-boundary value problems for higher-order layer profiles \((u^{1,j}, v^{1,j})\), \((u^{B,j+1}, v^{B,j})\) and \((u^{b,j+1}, v^{b,j})\) for \(j \geq 2\). But the equations (2.3)-(2.12) have been sufficient for our purpose. The detailed derivations of above equations are postponed to be given in Appendix, since it is a little lengthy. The global solutions of (2.3) have been achieved in [38] (see Lemma 2.1 below) and their regularities will be shown in section 3, while the existence of global solutions of (2.4)-(2.12) with regularities will be detailed in section 3.

2.2. Main results (stability of boundary layer profiles). In [38], the authors proved the global well-posedness of classical solutions to system (1.5)-(1.6) with \(\varepsilon \geq 0\). We cite the results below for later use.

**Lemma 2.1** ([38]). Suppose that \((u_0, v_0) \in H^2 \times H^2\) with \(u_0 \geq 0\) satisfy the compatibility conditions \((u_0, v_0)(0) = (u_0, v_0)(1) = (\bar{u}, \bar{v})\). Then for any \(\varepsilon \geq 0\), the initial-boundary value problem (1.5)-(1.6) has unique global classical solution \((u^\varepsilon, v^\varepsilon)\) satisfying the following properties:

(i) If \(\varepsilon > 0\), then \((u^\varepsilon - \bar{u}, v^\varepsilon - \bar{v}) \in C([0, \infty); H^2 \times H^2) \cap L^2(0, \infty; H^3 \times H^3)\) such that

\[
\| (u^\varepsilon - \bar{u})(t) \|_{L^2}^2 + \| (v^\varepsilon - \bar{v})(t) \|_{L^2}^2 + \int_0^t \| u_\varepsilon^\varepsilon(\tau) \|_{L^2}^2 + \varepsilon \| v_\varepsilon^\varepsilon(\tau) \|_{L^2}^2 \, d\tau \leq C,
\]

where \(C\) is a positive constant independent of \(\varepsilon\).

(ii) If \(\varepsilon = 0\), then \((u^0 - \bar{u}, v^0) \in C([0, \infty); H^2 \times H^2) \cap L^2(0, \infty; H^3 \times H^2)\).

In order to prove the stability of boundary layer solutions of (1.5)-(1.6), we need some further compatibility conditions on boundaries and higher regularity on the initial data \((u_0, v_0)\) to gain necessary estimates for solutions of equations (2.3)-(2.11). Precisely, we postulate that the initial data \((u_0, v_0) \in H^3 \times H^3\) satisfy

\[
(A) \quad \begin{cases} 
(u_0, v_0)|_{x=0,1} = (\bar{u}, \bar{v}), \\
u_{0x}|_{x=0,1} = 0, \\
[(u_0, v_0)x + u_{0xx}]|_{x=0,1} = 0.
\end{cases}
\]

We underline that the condition \((A)\) can be fulfilled by many functions, for instance \(u_0(x) = \bar{u} + ax^4(x-1)^4, v_0(x) = \bar{v} + bx^2(x-1)^2\) with \(a \geq 0\) and \(b \in \mathbb{R}\).

Now we are in a position to state the main results of this paper as follows.

**Theorem 2.1.** Assume that \((u_0, v_0) \in H^3 \times H^3\) with \(u_0 \geq 0\) satisfy the compatibility conditions \((A)\). Denote by \((u^\varepsilon, v^\varepsilon)\) the unique global solution of (1.5)-(1.6) with \(\varepsilon \geq 0\). Then as \(\varepsilon \to 0\), the following asymptotic expansions hold in space \(L^\infty([0, 1] \times [0, T])\) for any fixed \(0 < T < \infty\):

\[
u^\varepsilon(x, t) = u^0(x, t) + O(\varepsilon^{1/2}),
\]

\[
v^\varepsilon(x, t) = v^0(x, t) + v^{B,0}\left(\frac{x}{\sqrt{\varepsilon}}, t\right) + v^{b,0}\left(\frac{x-1}{\sqrt{\varepsilon}}, t\right) + O(\varepsilon^{1/2})
\]

where \((u^0, v^0) = (u^{1,0}, v^{1,0})\) denotes the outer layer profile and inner layer profile \((v^{B,0}, v^{b,0})\) is given by

\[
v^{B,0}(z, t) := \int_0^t \int_{-\infty}^0 \frac{1}{\pi(t-s)} e^{-\frac{(z-y)^2}{\pi(t-s)} + \bar{u}(t-s)} \left[ \bar{u}(v - v_0(0, s)) - v_0^0(s, 0) \right] dy ds,
\]

\[
v^{b,0}(x, t) := \int_0^t \int_{-\infty}^\infty \frac{1}{\pi(t-s)} e^{-\frac{(z-y)^2}{\pi(t-s)} + \bar{u}(t-s)} \left[ \bar{u}(v - v_0(1, s)) - v_0^0(1, s) \right] dy ds.
\]

**Remark 2.1.** We remark that (2.13) displays an elaborate structure of \(u^\varepsilon\) in the entire interval \([0, 1]\) including outer (approximated by \(v^0\)) and inner (approximated by \(v^{B,0}, v^{b,0}\)) boundary layer profiles, which significantly develops the result in previous work [26, Theorem 2.6], where only the outer layer profile for \(u^\varepsilon\) is obtained. The convergence rate \(O(\varepsilon^{1/2})\) in (2.13) is optimal, which improves the one \(O(\varepsilon^{1/4})\) derived in [26]. Furthermore the inner layer profile is explicitly connected to outer layer profile through (2.14).
A numerical simulation of the boundary layer solution component \( v^\varepsilon(x, t) \) is plotted in Fig. 1, where the structure of \( v^\varepsilon(x, t) \) is graphically demonstrated.

\[ \text{Figure 1. A numerical simulation of the boundary layer profile } v^\varepsilon(x, t) \text{ of the system (1.5)-(1.6) solved by the Matlab PDE solver based on the finite difference scheme with time step size } \Delta t = 0.01 \text{ and spatial step size } \Delta x = 0.001 \text{ where initial data } u_0(x) = 1 + x^4(x - 1)^4, v_0(x) = 1 + x^2(x - 1)^2 \text{ and boundary data } \bar{u} = \bar{v} = 1. \text{ The profile consists of two parts: outer layer profile } v^0 \text{ and inner layer profiles } v^{B,0} \text{ and } v^{b,0} \text{ near left and right end points, respectively. Outside the boundary layer the profile } v^\varepsilon(x, t) \text{ matches well with the outer layer profile } v^0, \text{ whereas there is a rapid transition inside the boundary layer.} \]

The counterpart of the original system (1.2) in \([0, 1]\) corresponding to the initial-boundary value problem of the transformed system (1.5)-(1.6) reads as follows:

\[
\begin{align*}
    u_t &= [u_x - u \ln(c)_x]_x, \\
    c_t &= \varepsilon c_{xx} - uc, \\
    (u, c)(x, 0) &= (u_0, c_0)(x), \quad x \in [0, 1], \\
    u|_{x=0,1} &= \bar{u}, \quad \frac{c_x}{c}|_{x=0,1} = -\bar{c}, \quad \text{if } \varepsilon > 0, \\
    u|_{x=0,1} &= \bar{u}, \quad \text{if } \varepsilon = 0.
\end{align*}
\] (2.15)

Denote by \((u^0, c^0)(x, t)\) the solution of (2.15) with \(\varepsilon = 0\). Then \(c^0(x, t)\) can be solved from the second equation \((\varepsilon = 0)\) of (2.15) as follows:

\[ c^0(x, t) = c_0(x)e^{\int_0^t u^0(x, \tau) \, d\tau}. \] (2.16)

With (2.16) and the results obtained for the transformed system (1.5)-(1.6), we have the following assertions for the initial-boundary value problem (2.15).

**Theorem 2.2.** Suppose that the initial data \((u_0, \ln c_0) \in H^3 \times H^3\) satisfy \(u_0(x) \geq 0, c_0(x) > 0\) and the compatibility conditions (A) with \(v_0 = -(\ln c_0)_x\) and \(\bar{v} = \bar{c}\). Let \((u^\varepsilon, c^\varepsilon)\) be the unique global solution of (2.15) with \(\varepsilon \geq 0\). Then for any fixed \(0 < T < \infty\), we have in space \(L^\infty([0, 1] \times [0, T])\) that

\[ u^\varepsilon(x, t) = u^0(x, t) + O(\varepsilon^{1/2}), \quad c^\varepsilon(x, t) = c^0(x, t) + O(\varepsilon^{1/2}) \] (2.17)
Compared with the previous result in [26, Proposition 2.8], Theorem 2.2 enhances the solution component $c_0(x, t)$ explicitly given via $u^0(x, t)$ in (2.16).

Remark 2.2. Compared with the previous result in [26, Proposition 2.8], Theorem 2.2 enhances the convergence rate by $\varepsilon^{1/4}$ and gives the leading-order expansion for $c_0^\varepsilon$ in (2.18), thanks to the elaborated approximation (2.13) for solutions $(u^\varepsilon, w^\varepsilon)$.

In view of model (1.2) and the transformation (1.3), we see that the quantity $v$ represents the velocity of chemotactic flux crossing the boundary (in the tumor angiogenesis the blood vessel wall can be understood as a boundary). Therefore the results in Theorem 2.2 assert that although both cell density and chemical concentration will have no boundary layer as chemical diffusion $\varepsilon$ goes to zero, the chemotactic flux, namely the term $u\ln c_x = -uw$, has a sharp transition near the boundary (i.e. the endothelial cells cross the blood vessel wall quickly). Hence our results indicate that the diffusion of chemical signal (i.e. vascular endothelial growth factor) plays an essential role in the transition of cell mass from boundaries to the field away from boundaries during the initiation of tumor angiogenesis. Our results further indicate that the non-diffusive model (2.15) with $\varepsilon = 0$ is not a good approximation of the diffusive model (2.15) for small $\varepsilon > 0$ near the boundary under the boundary conditions imposed in (2.15).

The rest of paper is organized as follows. In section 3, we shall derive some regularity of global solutions to (2.3)-(2.12). In section 4, we reformulate our problem properly and then prove Theorem 2.1 by the refined energy estimates based on the regularity results derived in section 3. The proof of Theorem 2.2 will be given in section 5. Section 6 is an appendix where we detail the asymptotical analysis of obtaining the equations (2.3)-(2.12).

3. Regularity of outer/inner layer profiles

In this section, we shall devote ourselves to deriving some regularities for solutions of (2.3)-(2.12) for later use. We depart with a basic regularity result.

Let functions $f_1(x, t)$, $f_2(x, t)$, $f(x, t)$ and $g(x, t)$ defined on $[0, 1] \times [0, \infty)$ satisfy the following regularity properties for any $m \in \mathbb{N}_+$ and $0 < T < \infty$:

$$\partial^k_x f_1 \in L^2(0, T; H^{2m-1-2k}), \quad \partial^k_x f_2 \in L^2(0, T; H^{2m-1-2k}),$$
$$\partial^k_x f \in L^2(0, T; H^{2m-2-2k}), \quad \partial^k_x g \in L^2(0, T; H^{2m-1-2k}),$$

where $k = 0, 1, \cdots, m - 1$. To solve the outer layer solution pairs $(u^{1,J}, v^{0,J})(x, t)$, $j = 0, 1$ from problems (2.3) and (2.8), we first consider the following auxiliary initial-boundary value problem

$$\begin{cases}
    h_t = (f_1 h)_x + (f_2 w)_x + h_{xx} + f, & (x, t) \in (0, 1) \times (0, T), \\
    w_t = h_x + g, & \\
    (h, w)(x, 0) = (h_0, w_0)(x), & \\
    h(0, t) = h(1, t) = 0.
\end{cases}$$

(3.1)

To derive the desired regularity (3.2) for solutions $(h, w)$ of (3.1) (see Proposition 3.1 below), some compatibility conditions on $h_0$, $w_0$, $f_1$, $f_2$, $f$ and $g$ are required. In the sequel, by $\"h_0, w_0, f_1, f_2, f, g\"$ we mean that $\partial^k_x h|_{x=0}$, which is determined by $h_0$, $w_0$, $f_1$, $f_2$, $f$, $g$ and their time derivatives through the equations in (3.1), are equal to zeros on boundaries for $0 \leq k \leq m - 1$ (cf. [33, page 319]).

Then the solution of (3.1) has the following regularity properties.

**Proposition 3.1.** Suppose that $(h_0, w_0) \in H^{2m-1} \times H^{2m-1}$, $f_1, f_2, f$ and $g$ satisfy the compatibility conditions up to order $(m - 1)$ for the problem (3.1). Then there exists a unique solution...
The proof of global existence and uniqueness of solutions to (3.1) is standard (see Lemma 2.1). The regularity given in (3.2) can be proved by mathematical induction. For $m = 1$, the conclusion follows from the standard energy method used in [38, Proof of Theorem 1.1] and we hence omit the details. The remaining procedure of mathematical induction is routine (e.g. see details in [12, page 387-388]) and will be skipped for brevity.

To solve inner layer profiles $v^{B,0}(z,t)$ and $v^{B,1}(z,t)$ from (2.4) and (2.9), we need the following result.

**Proposition 3.2.** Let $m \in \mathbb{N}_+$ and $0 < T < \infty$. Suppose $\rho(z,t)$ satisfies for any $l \in \mathbb{N}$ that

$$
\langle z \rangle^l \partial^k \rho \in L^2(0,T;H^{2m-2k}), \quad k = 0,1,\cdots,m-1,
$$

and the compatibility conditions up to order $(m-1)$ for the following problem:

$$
\begin{align*}
\varphi_t &= -\tilde{u}\varphi + \varphi_{zz} + \rho, & (z,t) \in (0,\infty) \times (0,T), \\
\varphi(z,0) &= 0, \\
\varphi(0,t) &= 0.
\end{align*}
$$

Then there exists a unique solution $\varphi$ to (3.3) such that for any $l \in \mathbb{N}$,

$$
\langle z \rangle^l \partial^k \varphi \in L^2(0,T;H^{2m-2k}), \quad k = 0,1,\cdots,m.
$$

Proposition 3.2 follows directly from the standard energy method, and we hence omit the proof. We proceed to introduce the following well-known result for later use.

**Proposition 3.3 ([62, Lemma 1.2.]).** Let $V, H, V'$ be three Hilbert spaces, satisfying $V \subset H \subset V'$ with $V'$ being the dual of $V$. If a function $u$ belongs to $L^2(0,T;V)$ and its time derivative $u_t$ belongs to $L^2(0,T;V')$, then

$$
u \in C([0,T];H) \quad \text{and} \quad \|u\|_{L^\infty(0,T;H)} \leq C(\|u\|_{L^2(0,T;V)} + \|u_t\|_{L^2(0,T;V')}),$$

where the constant $C$ depends on $T$.

**Remark 3.1.** Let $m \in \mathbb{N}$. Suppose that $u \in L^2(0,T;H^{m+2})$ and $u_t \in L^2(0,T;H^m)$. Then it follows from Proposition 3.3 that

$$
u \in C([0,T];H^{m+1}) \quad \text{and} \quad \|u\|_{L^\infty(0,T;H^{m+1})} \leq C(\|u\|_{L^2(0,T;H^{m+2})} + \|u_t\|_{L^2(0,T;H^m)}).$$

Based on above preliminaries, we can establish the regularities of solutions to (2.3)-(2.12). First for the problem (2.3), the existence of global solution has been available (see Lemma 2.1). We prove the following regularity results.

**Lemma 3.1.** Let $(u_0,v_0) \in H^3 \times H^3$ satisfy the assumptions in Theorem 2.1. Then the unique solution $(u^{I,0},v^{I,0})$ of (2.3) satisfies that

$$
\begin{align*}
\partial^k u^{I,0} &\in L^2(0,T;H^{1-2k}), \quad k = 0,1,2; \\
v^{I,0} &\in L^\infty(0,T;H^3); \quad \partial^k v^{I,0} \in L^2(0,T;H^{5-2k}), \quad k = 1,2.
\end{align*}
$$

**Proof.** We shall prove this lemma by Proposition 3.1 and Lemma 2.1. Differentiating the first and second equations of (2.3) with respect to $t$ respectively, and setting $\tilde{u}^{I,0} = u^{I,0}_t$, $\tilde{v}^{I,0} = v^{I,0}_t$, one gets

$$
\begin{align*}
\tilde{u}^{I,0}_t &= (f_1 \tilde{u}^{I,0})_x + (f_2 \tilde{v}^{I,0})_x + \tilde{u}^{I,0}_{xx}, \\
\tilde{v}^{I,0}_t &= \tilde{u}^{I,0}_x, \\
(\tilde{u}^{I,0}, \tilde{v}^{I,0})(x,0) &= (\tilde{u}_0, \tilde{v}_0)(x), \\
\tilde{u}^{I,0}(0,t) &= \tilde{u}^{I,0}(1,t) = 0.
\end{align*}
$$

(3.4)
where $f_{1} := v^{I,0}$, $f_{2} := u^{I,0}$, $\tilde{v}_{0} := (u_{0}v_{0})_{x} + u_{0xx}$, $\tilde{v}_{0} := u_{0x}$ and the first and second equations of (2.3) have been used to determine initial data $\tilde{u}_{0}$ and $\tilde{v}_{0}$, respectively. We next verify that $\tilde{u}_{0}$, $\tilde{v}_{0}$, $f_{1}$ and $f_{2}$ fulfill the assumptions in Proposition 3.1 with $m = 1$. First, by the assumptions in Theorem 2.1 and using Lemma 2.1 one finds that
\[
\|\tilde{u}_{0}\|_{H^{1}} + \|\tilde{v}_{0}\|_{H^{1}} \leq C_{0}; \quad \|f_{1}\|_{L^{2}(0,T;H^{1})} + \|f_{2}\|_{L^{2}(0,T;H^{1})} \leq C.
\] (3.5)

Noting that the compatibility condition of order zero for (3.4) is satisfied under assumption (A), thus using (3.5), we apply Proposition 3.1 with $m = 1$ to system (3.4) and conclude that
\[
\partial_{t}^{k} u^{I,0} \in L^{2}(0,T;H^{4-k}), \quad k = 1, 2; \quad \partial_{t} v^{I,0} \in L^{2}(0,T;H^{1}),
\] (3.6)
where $\tilde{u}^{I,0}$ and $\tilde{v}^{I,0}$ have been used. It only remains to prove
\[
u^{I,0} \in L^{2}(0,T;H^{4}), \quad v^{I,0} \in L^{\infty}(0,T;H^{3}), \quad v_{1}^{I,0} \in L^{2}(0,T;H^{3}).
\] (3.7)
To this end, we apply the differential operator $\partial_{z}^{2}$ to the second equation of (2.3), and use the first equation of (2.3) to get
\[
\tilde{v}_{0}^{I,0} = u^{I,0} - (u^{I,0}v^{I,0})_{xxx},
\] (3.8)
which, multiplied by $2\tilde{v}_{0}^{I,0}$ in $L^{2}$ gives
\[
\frac{d}{dt}\|\tilde{v}_{0}^{I,0}(t)\|_{L^{2}} \leq 2\|\tilde{v}_{0}^{I,0}(t)\|_{L^{2}} + C_{0}\|u^{I,0}(t)\|_{H^{3}}\|v^{I,0}(t)\|_{H^{3}}^{2}
\]
\[
\leq C_{0}(1 + \|u^{I,0}(t)\|_{H^{3}})\|\tilde{v}_{0}^{I,0}(t)\|_{L^{2}}^{2} + C_{0}\|u_{t}^{I,0}(t)\|_{H^{2}} + \|u^{I,0}(t)\|_{H^{3}}\|v^{I,0}(t)\|_{H^{2}}^{2}.
\]
Thus it follows from Gronwall’s inequality, Lemma 2.1 and (3.6) that
\[
\|\tilde{v}_{0}^{I,0}\|_{L^{\infty}(0,T;L^{2})} \leq C.
\] (3.9)
Furthermore, using (3.8), (3.6), (3.9) and Lemma 2.1, one has
\[
\|u_{x}^{I,0}\|_{L^{2}(0,T;H^{2})} \leq \|u_{x}^{I,0}\|_{L^{2}(0,T;H^{2})} + C_{0}\|u^{I,0}\|_{L^{2}(0,T;H^{3})}\|v^{I,0}\|_{L^{\infty}(0,T;H^{3})} \leq C.
\] (3.10)
Finally, the second equation of (2.3) along with (3.10) and Lemma 2.1 yields
\[
\|\tilde{v}_{1}^{I,0}\|_{L^{2}(0,T;H^{3})} \leq \|v^{I,0}\|_{L^{2}(0,T;H^{4})} \leq C.
\] (3.11)
Collecting (3.9), (3.10), (3.11) and using Lemma 2.1 we obtain (3.7), which in conjunction with (3.6) finishes the proof.

\[\square\]

Lemma 3.2. Let $(u^{I,0}, v^{I,0})$ be the solution obtained in Lemma 3.1. Then
\[
v^{B,0}(z,t) := \int_{0}^{t} \int_{0}^{t} \frac{1}{\sqrt{\pi(t-s)}} e^{-\left(\frac{(z-y)^{2}}{4(t-s)}+\tilde{v}(t-s)\right)} \left[\tilde{u}(\tilde{v}-v^{I,0}(s)) - v_{x}^{I,0}(s)\right] dy ds
\] (3.12)
is the unique solution of (2.4). Moreover, for any $0 < T < \infty$ and $l \in \mathbb{N}$, it holds that
\[
\langle z \rangle^{k} \partial_{t}^{l} v^{B,0} \in L^{2}(0,T;H^{4-k}_{z})
\] (3.13)
for $k = 0, 1, 2$.

Consequently it follows from (2.5) that
\[
\langle z \rangle^{k} \partial_{t}^{l} u^{B,1} \in L^{2}(0,T;H^{4-k}_{z})
\] (3.14)
for $k = 0, 1, 2$.

Proof. We first prove (3.12) by setting $w(z,t) := e^{\tilde{v}t}[v^{B,0}(z,t) - (\tilde{v} - v^{I,0}(0,t))]$. Then from (2.4) we derive the following heat equation subject to homogeneous Dirichlet boundary condition
\[
\begin{cases}
w_{z} - w_{zz} = -[e^{\tilde{v}t}(\tilde{v} - v^{I,0}(0,t))]_{t}, & (z,t) \in (0,\infty) \times (0,\infty) \\
w(z,0) = 0, \\
w(0,t) = 0,
\end{cases}
\]
which can be solved explicitly by the reflection method with odd extensions (cf. [33]) as follows:
\[
w(z,t) = 2 \int_{0}^{t} \int_{-\infty}^{0} \Gamma(z-y, t-s) [e^{\tilde{u}s}(\tilde{v} - v^{I,0}(0,s))]_{s} dy ds - e^{\tilde{u}t}(\tilde{v} - v^{I,0}(0,t)),
\]
with the heat kernel $\Gamma(z, t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{z^2}{4t}}$. Hence (3.12) follows by substituting the above equality into the definition of $w(z, t)$. We proceed to prove (3.13). Let $\theta(z)$ be a smooth function defined on $[0, \infty)$ satisfying
\[
\theta(0) = 1, \quad \theta(z) = 0 \quad \text{for } z > 1.
\] (3.14)
Let $b(t) := \bar{v} - v^{I,0}(0, t)$ and $\bar{v}^{B,0} := v^{B,0} - \theta(z)b(t)$. Then from (2.4) we deduce that $\bar{v}^{B,0}$ satisfies
\[
\begin{cases}
\frac{\partial}{\partial t} \bar{v}^{B,0} = -\bar{v}^{B,0} + \bar{v}_{zz}^B + \bar{\rho}, \\
\bar{v}^{B,0}(z, 0) = 0, \\
\bar{v}^{B,0}(0, t) = 0,
\end{cases}
\]
(3.15)
where $\bar{\rho}(z, t) := \theta_{zz}(z)b(t) - \bar{u}\theta(z)b(t) - \theta(z)b_{0}(t)$, and the compatibility condition $\bar{v} = v_{0}(0)$ has been used to determine the initial value of $\bar{v}^{B,0}$. We shall apply Proposition 3.2 to (3.15) to derive the desired regularity for $v^{B,0}$. To this end, we need to verify that $\bar{\rho}$ satisfies the assumptions in Proposition 3.2 with $m = 2$. First, it is easy to check that $\bar{\rho}$ satisfies the compatibility conditions up to order one for problem (3.15) under assumption (A). Then noticing that for any $G(x, t) \in L^p(0, T; H^1)$ with $1 \leq p \leq \infty$, it follows from the Sobolev embedding inequality that
\[
\|G(0, t)\|_{L^{p}(0, T)} \leq \|G\|_{L^{p}(0, T; L^{\infty})} \leq C_{0}\|G\|_{L^{p}(0, T; H^{1})}.
\]
(3.16)
By (3.16) and Lemma 3.1, one finds that
\[
\|\partial_{t}^{k}\bar{v}^{I,0}(0, t)\|_{L^{2}(0, T)} \leq C, \quad k = 1, 2; \quad \|v^{I,0}(0, t)\|_{L^{2}(0, T)} \leq C.
\]
(3.17)
Collecting (3.14) and (3.17), one deduces for $k = 0, 1$ and $l \in \mathbb{N}$ that
\[
\langle z \rangle^{l} \partial_{l}^{k} \bar{\rho} = \langle z \rangle^{l} \theta_{zz} \partial_{l}^{k} b - \bar{u} \langle z \rangle^{l} \theta \partial_{l}^{k+1} b \in L^{2}(0, T; H^{2-2k}_{z}),
\]
which, along with Proposition 3.2 entails for $k = 0, 1, 2$ and $l \in \mathbb{N}$ that
\[
\langle z \rangle^{l} \partial_{l}^{k} \bar{v}^{B,0} \in L^{2}(0, T; H^{2-2k}_{z}).
\]
Thus (3.13) follows from the definition of $\bar{v}^{B,0}$, (3.14) and (3.17). Finally by (3.13), we use (2.5) and Hölder inequality to get for $k = 0, 1, 2$ and $l \in \mathbb{N}$ that
\[
\|\langle z \rangle^{l} \partial_{l}^{k} u^{B,1}\|^{2}_{L^{2}(0, T; H^{2-2k}_{z})} \leq C_{0} \bar{u}^{2}
\]
(3.18)
\[
+ \int_{z}^{\infty} \int_{0}^{\infty} (y)^{-4} dy dz \|\langle z \rangle^{l+2} \partial_{l}^{k} v^{B,0}\|^{2}_{L^{2}(0, T; H^{4-2k}_{z})} \leq C,
\]
which completes the proof.

By a similar procedure as proving Lemma 3.2, we have the following results.

**Lemma 3.3.** Let $(u^{I,0}, v^{I,0})$ be the solution obtained in Lemma 3.1. Then the unique solution $v^{b,0}(\xi, t)$ of (2.6) is as follows:
\[
v^{b,0}(\xi, t) := \int_{0}^{t} \int_{0}^{\infty} \frac{1}{\sqrt{\pi(t-s)}} e^{-\frac{\xi^2}{4(t-s)}} \left[ \bar{u}(\bar{v} - v^{I,0}(1, s)) - v^{I,0}(1, s) \right] dy ds.
\]
(3.19)
Furthermore, for any $0 < T < \infty$ and $l \in \mathbb{N}$, the following holds true:
\[
\langle \xi \rangle^{l} \partial_{l}^{k} v^{b,0}, \quad \langle \xi \rangle^{l} \partial_{l}^{k} u^{b,1} \in L^{2}(0, T; H^{2-2k}_{\xi}) \quad \text{for } k = 0, 1, 2.
\]

Based on Lemma 3.2 and Lemma 3.3, we proceed to solve (2.8).

**Lemma 3.4.** Let $v^{B,0}$ and $v^{b,0}$ be the solution obtained in Lemma 3.2 and Lemma 3.3, respectively. Then there exists a unique solution $(u^{I,1}, v^{I,1})$ to (2.8) on $[0, T]$ for any $0 < T < \infty$, such that
\[
\partial_{t}^{k} u^{I,1} \in L^{2}(0, T; H^{1-2k}_{\xi}), \quad k = 0, 1, 2; \quad v^{I,1} \in L^{\infty}(0, T; H^{3}), \quad \partial_{t}^{k} v^{I,1} \in L^{2}(0, T; H^{5-2k}_{\xi}), \quad k = 1, 2.
Proof. Let \( b_1(t) := \tilde{u} \int_0^\infty v^{B,0}(z, t) \, dz \), \( b_2(t) := \tilde{u} \int_0^\infty v^{b,0}(\xi, t) \, d\xi \), \( b(x, t) := xb_2(t) + (1 - x)b_1(t) \) and \( \tilde{u}^{l,1} := u^{l,1} + b(x, t) \). Then from (2.8), we deduce that \( (\tilde{u}^{l,1}, v^{l,1}) \) satisfy

\[
\begin{aligned}
\tilde{u}_{t}^{l,1} &= (f_1 \tilde{u}^{l,1})_x + (f_2 v^{l,1})_x + \tilde{u}_{xx}^{l,1} + f, \\
v_{t}^{l,1} &= \tilde{u}_{x}^{l,1} + g, \\
(\tilde{u}^{l,1}, v^{l,1})(x, 0) &= (0, 0), \\
\tilde{u}^{l,1}(0, t) = u^{l,1}(1, t) = 0,
\end{aligned}
\tag{3.20}
\]

where \( f_1 := v^{l,0}, f := -(b v^{l,0})_x + b_1, g := b_1(t) - b_2(t) \), and \( v^{B,0}(z, 0) = v^{b,0}(\xi, 0) = 0 \) has been used in deriving the initial data for \( \tilde{u}^{l,1} \). We next verify that \( f_1, f, g \) and \( f \) fulfill the assumptions in Proposition 3.1 with \( m = 2 \). Indeed, it follows from Lemma 3.1 that

\[
\partial_t^k v^{l,0} \in L^2(0, T; H^{2-2k}), \quad k = 0, 1.
\tag{3.21}
\]

Lemma 3.2 gives for \( k = 0, 1, 2 \) that

\[
||\partial_t^k b_1||_{L^2(0, T)} \leq \tilde{u}^2 \int_0^\infty \langle z \rangle^{-2} dy \cdot ||\langle z \rangle \partial_t^k v^{B,0}||_{L^2(0, T; L^2)} \leq C
\tag{3.22}
\]

and similarly Lemma 3.3 implies for \( k = 0, 1, 2 \) that

\[
||\partial_t^k b_2||_{L^2(0, T)} \leq C.
\tag{3.23}
\]

Thus from (3.22), (3.23) and the definition of \( g \), we have

\[
\partial_t^k g \in L^2(0, T; H^{2-2k}), \quad k = 0, 1.
\tag{3.24}
\]

To estimate \( f \), we use (3.22)-(3.23), Lemma 3.1 and Proposition 3.3 and get for \( k = 0, 1 \) that

\[
||\partial_t^k (bv^{l,0})_x||_{L^2(0, T; H^{2-2k})} \leq \sum_{j=0}^k \left( ||\partial_t^j b_1||_{L^2(0, T)} + ||\partial_t^j b_2||_{L^2(0, T)} \right) ||\partial_t^{k-j} v^{l,0}||_{L^\infty(0, T; H^{2-2(k-j)})} \leq C,
\]

which, in conjunction with the definition of \( f \), (3.22) and (3.23) entails that

\[
\partial_t^k f \in L^2(0, T; H^{2-2k}), \quad k = 0, 1.
\tag{3.25}
\]

Noting that for (3.20), compatibility conditions up to order one are fulfilled under assumption (A), thus by (3.21), (3.24) and (3.25), we apply Proposition 3.1 with \( m = 2 \) to (3.20) and get

\[
\begin{aligned}
\partial_t^k \tilde{u}^{l,1} &\in L^2(0, T; H^{4-2k}), \quad k = 0, 1, 2; \\
v^{l,1} &\in L^\infty(0, T; H^1); \quad \partial_t^k v^{l,1} \in L^2(0, T; H^{5-2k}), \quad k = 1, 2.
\end{aligned}
\tag{3.26}
\]

The first estimate in (3.26) along with the definition of \( \tilde{u}^{l,1} \), (3.22) and (3.23) gives rise to

\[
\partial_t^k u^{l,1} \in L^2(0, T; H^{4-2k}), \quad k = 0, 1, 2.
\tag{3.27}
\]

Thus the combination of (3.26) and (3.27) completes the proof.

We next turn to the regularity of solutions to (2.9) and (2.10):

**Lemma 3.5.** Let \((u^{l,1}, v^{l,1})\) be the solution obtained in Lemma 3.4. Then there exists a unique solution \( v^{B,1} \) to (2.9) on \([0, T]\) for any \( 0 < T < \infty \), such that for any \( l \in \mathbb{N} \),

\[
\langle z \rangle^l \partial_t^k v^{B,1} \in L^2(0, T; H^{2-2k}_z), \quad k = 0, 1, 2.
\tag{3.28}
\]

Consequently, it follows from (2.10) that

\[
\langle z \rangle^l \partial_t^k u^{B,2} \in L^2(0, T; H^{4-2k}_z), \quad k = 0, 1.
\tag{3.29}
\]
Proof. Let \( \tilde{v}^{B,1} := v^{B,1} + \theta(z)v^{f,1}(0, t) \) with \( \theta \) defined in (3.14). Then from (2.9), we deduce that \( \tilde{v}^{B,1} \) satisfies

\[
\begin{cases}
\partial_t \tilde{v}^{B,1} = -\tilde{u}^{B,1} + \tilde{v}^{B,1}_{zz} + \rho, \\
\tilde{v}^{B,1}(z, 0) = 0, \\
\tilde{v}^{B,1}(0, t) = 0,
\end{cases}
\]

where \( \rho := \tilde{u}\theta v^{f,1}(0, t) + \theta v^{l,1}(0, t) - \theta z z v^{f,1}(0, t) - 2(v^{f,0}(0, t) + v^{B,0})v^{B,0} + \int_{z}^{\infty} \Phi(y, t) \, dy. \) We shall apply Proposition 3.2 with \( m = 2 \) to (3.30) to prove this lemma by verifying that \( \rho \) satisfies the assumptions in Proposition 3.2. Let us start by dividing \( \rho \) into three parts:

\[
\rho = \tilde{u}\theta v^{f,1}(0, t) + \theta v^{l,1}(0, t) - \theta z z v^{f,1}(0, t) - 2(v^{f,0}(0, t) + v^{B,0})v^{B,0} + \int_{z}^{\infty} \Phi(y, t) \, dy
\]

(3.31)

\[:= I_1 + I_2 + I_3.\]

We next estimate \( I_1, I_2 \) and \( I_3. \) First it follows from (3.16) and Lemma 3.4 that

\[
\| \partial_t^k v^{f,1}(0, t) \|_{L^2(0, T)} \leq C_0 \| \partial_t^k v^{f,1} \|_{L^2(0, T; H^4)} \leq C, \quad k = 0, 1, 2,
\]

which, along with the definition of \( \theta \) in (3.14) implies that

\[
\langle z \rangle \partial_t^k I_1 \in L^2(0, T; H_z^{2-2k}), \quad l \in \mathbb{N}, \quad k = 0, 1.
\]

(3.32)

Then applying (3.16) to \( \partial_t^k v^{f,0} \) and using Lemma 3.1, Lemma 3.2 and Remark 3.1, we have for \( k = 0, 1 \) and \( l \in \mathbb{N} \) that

\[
\| \partial_t^k I_2 \|_{L^2(0, T; H_z^{2-2k})} \leq C_0 \sum_{j=0}^{k} (\| \partial_t^j v^{f,0} \|_{L^\infty(0, T; H_z^{2-2j})} + \| \partial_t^j v^{B,0} \|_{L^\infty(0, T; H_z^{2-2j})}) \| \partial_t^{k-j} v^{B,0} \|_{L^2(0, T; H_z^{2-2(k-j)})}
\]

(3.33)

\[
\leq C.
\]

For \( I_3 \), the estimate is a little more complicated, since it involves several terms. The Hölder inequality entails for \( k = 0, 1, 1 \) and \( l \in \mathbb{N} \) that

\[
\| \langle z \rangle \partial_t^k I_3 \|_{L^2(0, T; H_z^{2-2k})} \leq C_0 \left( 1 + \int_0^\infty \int_{z}^{\infty} (y)^{-4} \, dy \, dz \right) \| \langle z \rangle^{l+2} \partial_t^k \Phi \|_{L^2(0, T; H_z^{2-2k})}.
\]

(3.34)

Noting that the integration term in parentheses of the above inequality is finite, we only need to estimate the remaining term. By the definition of \( \Phi \) below (2.10), one gets for \( l \in \mathbb{N} \) that

\[
\langle z \rangle \partial_t^k \Phi = \langle z \rangle \partial_t^k [u^{f,1}(0, t)v^{B,0}] + \langle z \rangle \partial_t^k [u^{B,1}v^{B,0}] + \langle z \rangle \partial_t^k [u^{f,0}(0, t)v^{B,0}] + z \langle z \rangle \partial_t^k [u^{f,0}(0, t)v^{B,0}]
\]

(3.35)

\[:= M_1 + M_2 + M_3 + M_4 + M_5 + M_6.
\]

Applying (3.16) to \( \partial_t^k v^{f,1} \), by Lemma 3.2, Lemma 3.4 and Remark 3.1, we get for \( k = 0, 1 \), that

\[
\| M_1 \|_{L^2(0, T; H_z^{2-2k})} \leq C_0 \sum_{j=0}^{k} \| \partial_t^j v^{f,1} \|_{L^\infty(0, T; H_z^{2-2j})} \| \langle z \rangle \partial_t^{k-j} v^{B,0} \|_{L^2(0, T; H_z^{2-2(k-j)})} \leq C.
\]

Similar arguments further give the estimate for \( \{ M_i \}_{2 \leq i \leq 6} \):

\[
\| M_i \|_{L^2(0, T; H_z^{2-2k})} \leq C, \quad 2 \leq i \leq 6, \quad k = 0, 1.
\]

Plugging the above estimates into (3.35), we conclude for any \( l \in \mathbb{N} \) that

\[
\langle z \rangle \partial_t^k \Phi \in L^2(0, T; H_z^{2-2k}), \quad k = 0, 1,
\]

(3.36)

which, along with (3.34) gives rise to

\[
\langle z \rangle \partial_t^k I_3 \in L^2(0, T; H_z^{2-2k}), \quad k = 0, 1.
\]

(3.37)

Then it follows from (3.31), (3.32), (3.33) and (3.37) that

\[
\langle z \rangle \partial_t^k \rho \in L^2(0, T; H_z^{2-2k}), \quad k = 0, 1, \quad l \in \mathbb{N}.
\]
Moreover for (3.30) it is easy to check that $\rho$ fulfills the compatibility conditions up to order one under assumption (A). Thus by (3.38), we apply Proposition 3.2 with $m = 2$ to (3.30) and have

$$\langle z \rangle^l \partial^k_z v^{B.1} \in L^2(0, T; H^2_{-2k}), \quad k = 0, 1, 2, \ l \in \mathbb{N}. \quad (3.39)$$

To convert the result in (3.39) back to $v^{B.1}$, we note that

$$\langle z \rangle^l \partial^k_z v^{B.1} = \langle z \rangle^l \partial^k_z \tilde{v}^{B.1} - \langle z \rangle^l \theta(z) \partial^k_z v^{J.1}(0, t), \quad (3.40)$$

where the second term on the right-hand side is estimated by the definition of $\theta$, (3.16) and Lemma 3.4 for $k = 1, 2$ and $l \in \mathbb{N}$ as:

$$\|\langle z \rangle^l \theta(z) \partial^k_z v^{J.1}(0, t)\|_{L^2(0, T; H^2_{-2k})} \leq C_0 \|\partial^k_z v^{J.1}(0, t)\|_{L^2(0, T; H^2_{-2k})} \leq C_0 \|\partial^k_z v^{J.1}\|_{L^2(0, T; H^2_{-2k})} \leq C \quad \text{and for } k = 0 \text{ and } l \in \mathbb{N} \text{ as:}$$

$$\|\langle z \rangle^l \theta(z) v^{J.1}(0, t)\|_{L^2(0, T; H^2_{-2k})} \leq C_0 \|v^{J.1}(0, t)\|_{L^2(0, T)} \leq C \|v^{J.1}\|_{L^\infty(0, T; H^2_{-2k})} \leq C.$$

Inserting the above two estimates with (3.39) into (3.40), we derive (3.28). It remains to estimate $u^{B.2}$ and for $k \neq 0$ and $l \in \mathbb{N}$ as:

$$\|\langle z \rangle^l \theta(z) v^{J.1}(0, t)\|_{L^2(0, T; H^2_{-2k})} \leq C_0 \|v^{J.1}(0, t)\|_{L^2(0, T)} \leq C \|v^{J.1}\|_{L^\infty(0, T; H^2_{-2k})} \leq C.$$

Substituting the above estimates for $I_4$ and $I_5$ into (3.41) one gets (3.29). The proof is completed.

Noticing the similarity between (2.9) and (2.11), by analogous arguments as proving Lemma 3.5, one gets that

**Lemma 3.6.** Let $(u^{J.1}, v^{J.1})$ be the solution obtained in Lemma 3.4. Then there exists a unique solution $v^{b.1}$ to (2.11) on $[0, T]$ for any $0 < T < \infty$, such that for any $l \in \mathbb{N}$,

$$\langle \xi \rangle^l \partial^k_z v^{b.1} \in L^2(0, T; H^2_{-2k}), \quad k = 0, 1, 2,$$

and

$$\langle \xi \rangle^l \partial^k_z u^{b.2} \in L^2(0, T; H^2_{-2k}), \quad k = 0, 1.$$

4. **Stability of Boundary Layers (Proof of Theorem 2.1)**

4.1. **Reformulation of the problem.** To prove Theorem 2.1, if we decompose the solution $(u^\varepsilon, v^\varepsilon)$ as:

$$u^\varepsilon(x, t) = u^{J.0}(x, t) + R^\varepsilon_1(x, t),$$

$$v^\varepsilon(x, t) = v^{J.0}(x, t) + v^{B.0}\left(\frac{x}{\varepsilon}, t\right) + v^{b.0}\left(\frac{x - 1}{\sqrt{\varepsilon}}, t\right) + R^\varepsilon_2(x, t), \quad (4.1)$$

then it remains to derive the equations satisfied by $R^\varepsilon_i(x, t) (i = 1, 2)$, and to show

$$\|R^\varepsilon_i\|_{L^\infty((0, 1) \times [0, T])} = \mathcal{O}(\varepsilon^{1/2}).$$

But if we substitute (4.1) into equations (1.5), we shall find that the equations of $R^\varepsilon_i$ have source terms containing a singular quantity of order $\varepsilon^{-1/2}$, which brings difficulties to derive the uniform-in-$\varepsilon$ boundedness of $\|R^\varepsilon_i\|_{L^\infty((0, 1) \times [0, T])} (i = 1, 2)$. Therefore we invoke the higher order
terms in the expansion of \((u^\varepsilon, v^\varepsilon)\) to overcome this difficulty motivated by a work [47]. This end, we employ (2.1)-(2.2) to write \(R_i^\varepsilon(x,t)(i=1,2)\) as:
\[
R_1^\varepsilon(x,t) = \varepsilon^{1/2}[u^{I,1}(x,t) + u^{B,1}(z,t) + u^{b,1}(\xi,t)] + \varepsilon[u^{B,2}(z,t) + u^{b,2}(\xi,t)] \\
+ \tilde{b}_1^\varepsilon(x,t) + \varepsilon^{1/2}U^\varepsilon(x,t),
\]
\[
R_2^\varepsilon(x,t) = \varepsilon^{1/2}[v^{I,1}(x,t) + u^{B,1}(z,t) + v^{b,1}(\xi,t)] \\
+ \tilde{b}_2^\varepsilon(x,t) + \varepsilon^{1/2}V^\varepsilon(x,t),
\]
where the perturbation functions \((U^\varepsilon, V^\varepsilon)(x,t)\) are to be determined, and the auxiliary functions \(b_i(x,t)(i=1,2)\) are constructed as follows to homogenize the boundary conditions of \((U^\varepsilon, V^\varepsilon)(x,t)\):
\[
b_1^\varepsilon(x,t) = -(1-x)\varepsilon^{1/2}u^{b,1}(-\varepsilon^{-1/2},t) + \varepsilon u^{B,2}(0,t) + \varepsilon u^{b,2}(-\varepsilon^{-1/2},t) \\
- x[\varepsilon^{1/2}u^{B,1}(\varepsilon^{-1/2},t) + \varepsilon u^{b,2}(0,t) + \varepsilon u^{B,2}(\varepsilon^{-1/2},t)],
\]
\[
b_2^\varepsilon(x,t) = -(1-x)[v^{b,0}(-\varepsilon^{-1/2},t) + \varepsilon^{1/2}v^{b,1}(-\varepsilon^{-1/2},t)] \\
- x[v^{B,0}(\varepsilon^{-1/2},t) + \varepsilon^{1/2}v^{B,1}(\varepsilon^{-1/2},t)].
\]

We should remark that the term \(u^{I,2}\) has been intentionally omitted in the expression of \(R_1^\varepsilon(x,t)\) since we find it is unnecessary for our purpose. Indeed if we include the term \(u^{I,2}\) in \(R_1^\varepsilon(x,t)\), then a higher regularity \(L^2(0,T;H^1)\) will be required on \(u^{I,2}\) in the proof of Lemma 4.1 when estimating \(f^\varepsilon\). This demands a higher regularity on initial data \((u_0, v_0)\) so that \((u_0, v_0)\) \(\in H^3 \times H^3\). Therefore, to reduce the regularity of \((u_0, v_0)\), we deliberately omit \(u^{I,2}\) in \(R_1^\varepsilon(x,t)\), which is a trick we employed.

For simplicity of presentation, with \(z\) and \(\xi\) given in (2.2) we define new functions
\[
\tilde{U}^\varepsilon(x,t) := u^{0,0}(x,t) + \varepsilon^{1/2}[u^{I,1}(x,t) + u^{B,1}(z,t) + u^{b,1}(\xi,t)] \\
+ \varepsilon[u^{B,2}(z,t) + u^{b,2}(\xi,t)] + \tilde{b}_1^\varepsilon(x,t),
\]
\[
\tilde{V}^\varepsilon(x,t) := v^{0,0}(x,t) + v^{B,0}(z,t) + v^{b,0}(\xi,t) \\
+ \varepsilon^{1/2}[v^{I,1}(x,t) + v^{B,1}(z,t) + v^{b,1}(\xi,t)] + \tilde{b}_2^\varepsilon(x,t),
\]
and then the perturbation functions \((U^\varepsilon, V^\varepsilon)(x,t)\) can be written as
\[
U^\varepsilon = \varepsilon^{-1/2}(u^\varepsilon - \tilde{U}^\varepsilon), \quad V^\varepsilon = \varepsilon^{-1/2}(v^\varepsilon - \tilde{V}^\varepsilon). \tag{4.2}
\]

Substituting (4.2) into (1.5)-(1.6) and using the initial-boundary conditions in (2.3)-(2.11), one finds that \((U^\varepsilon, V^\varepsilon)\) satisfies
\[
\begin{cases}
U_\varepsilon^t = \varepsilon^{1/2}(U^\varepsilon V^\varepsilon)_x + (U^\varepsilon \tilde{V}^\varepsilon)_x + (V^\varepsilon \tilde{U}^\varepsilon)_x + U_\varepsilon^\xi + \varepsilon^{-1/2}f^\varepsilon, \\
V_\varepsilon^t = -2\varepsilon^{3/2}V^\varepsilon V^\varepsilon_x - 2\varepsilon(V^\varepsilon \tilde{V}^\varepsilon)_x + U_\varepsilon^\xi + \varepsilon V_\varepsilon^\xi + \varepsilon^{-1/2}g^\varepsilon, \\
(U^\varepsilon, V^\varepsilon)(x,0) = (0,0), \\
(U^\varepsilon, V^\varepsilon)(0,t) = (U^\varepsilon, V^\varepsilon)(1,t) = (0,0),
\end{cases} \tag{4.3}
\]
with
\[
f^\varepsilon = \tilde{U}_\varepsilon^\xi + (\tilde{U}^\varepsilon \tilde{V}^\varepsilon)_x - \tilde{U}^\varepsilon, \quad g^\varepsilon = \varepsilon \tilde{V}_\varepsilon^\xi + \tilde{U}_\varepsilon^\xi - \varepsilon \tilde{U}^\varepsilon \tilde{V}^\varepsilon_x. \tag{4.4}
\]
Now the key is to give the \(L^\infty\)-estimates for the solution \((U^\varepsilon, V^\varepsilon)\) of (4.3)-(4.4), which will be gradually achieved in the sequel by the method of energy estimates.

4.2. Energy estimates. We shall develop various delicate energy estimates in this subsection to attain the \(L^\infty\) estimates of \((U^\varepsilon, V^\varepsilon)\) to (4.3)-(4.4). Before proceeding, we introduce some basic facts for later use. First for any \(G_1(z,t) \in H^m_z\) and \(G_2(\xi,t) \in H^m_\xi\) with \(m \in \mathbb{N}\), we have from the change of variables in (2.2) that
\[
\left\| \partial_x^m G_1 \left( \frac{x}{\sqrt{\varepsilon}}, t \right) \right\|_{L^2} = \varepsilon^{\frac{m}{2}} \left\| \partial_x^m G_1(z,t) \right\|_{L^2}, \tag{4.5}
\]
and
\[
\left\| \partial_x^{m} G_2 \left( \frac{x}{\sqrt{\varepsilon}}, t \right) \right\|_{L^2} = \varepsilon^{\frac{1-m}{2}} \left\| \partial_x^{m} G_2 (\xi, t) \right\|_{L^2_x} .
\]

(4.6)

For \( h(\cdot, t) \in H^1 \) with \( h|_{x=0,1} = 0 \), we have \( h^2(x, t) = 2 \int_0^x h h_y \, dy \leq 2 \| h(\cdot, t) \|_{L^2} \| h_x(\cdot, t) \|_{L^2} \). Thus
\[
\| h(\cdot, t) \|_{L^\infty} \leq \sqrt{2} \| h(\cdot, t) \|_{L^2} \| h_x(\cdot, t) \|_{L^2} \quad \text{and} \quad \| h(\cdot, t) \|_{L^\infty} \leq C_0 \| h_x(\cdot, t) \|_{L^2} .
\]

(4.7)

thanks to the Poincaré inequality \( \| h(\cdot, t) \|_{L^2} \leq C_0 \| h_x(\cdot, t) \|_{L^2} \).

We start with estimating \( f^\varepsilon \) and \( g^\varepsilon \).

**Lemma 4.1.** Let \( 0 < T < \infty, 0 < \varepsilon < 1 \) and \( f^\varepsilon \) be as defined in (4.4). Then there is a constant \( C \) independent of \( \varepsilon \), such that
\[
\| f^\varepsilon \|_{L^2(0, T; L^2)} \leq C \varepsilon^{3/4}.
\]

(4.8)

**Proof.** First applying the definitions of \( \tilde{U}^\varepsilon \) and \( \tilde{V}^\varepsilon \) into the expression of \( f^\varepsilon \) in (4.4) and using the first equations in (2.3) and in (2.8), we end up with
\[
f^\varepsilon = \varepsilon^{1/2} u_{xx}^{B,1} + \varepsilon^{1/2} u_{x}^{b,1} + \varepsilon u_{xx}^{B,2} + \varepsilon u_{xx}^{b,2} + \varepsilon (u^{I,1} u^{I,1})_x
\]
\[
+ \left[ (u^{I,0} + \varepsilon u^{I,1})(u^{B,0} + \varepsilon u^{b,0} + \varepsilon^{1/2} u^{B,1} + \varepsilon^{1/2} u^{b,1}) \right]_x
\]
\[
+ \left[ (\varepsilon^{1/2} u^{B,1} + \varepsilon u^{b,1} + \varepsilon u^{B,2} + \varepsilon u^{b,2}) \right]_x
\]
\[
\times (u^{I,0} + \varepsilon u^{B,0} + \varepsilon u^{b,0} + \varepsilon^{1/2} u^{B,1} + \varepsilon^{1/2} u^{b,1})_x
\]
\[
- \varepsilon u_{I}^{B,1} - \varepsilon u_{I}^{b,1} - \varepsilon u_{I}^{B,2} - \varepsilon u_{I}^{b,2} + F^\varepsilon,
\]

(4.9)

where
\[
F^\varepsilon := [b_1^\varepsilon (u^{I,0} + \varepsilon u^{B,0} + \varepsilon u^{b,0} + \varepsilon^{1/2} u^{B,1} + \varepsilon^{1/2} u^{b,1})]_x
\]
\[
+ [b_2^\varepsilon (u^{I,0} + \varepsilon u^{B,1} + \varepsilon u^{I,1} + \varepsilon u^{B,2} + \varepsilon u^{b,2})]_x + (b_1^\varepsilon b_2^\varepsilon)_x - b_1^\varepsilon .
\]

(4.10)

By the transformation (2.2), one gets from (6.8), (6.9), (6.12) and (6.13) (see Appendix) that
\[
\varepsilon u_{I}^{B,1} = - u^{I,0}(0,t) v_{x}^{B,0} ,
\]
\[
\varepsilon u_{I}^{B,2} = - x u^{I,0}(0,t) v_{x}^{B,0} - \varepsilon^{1/2} u^{I,1}(0,t) v_{x}^{B,0} - u^{I,0}(0,t) v_{x}^{B,0}
\]
\[
- \varepsilon^{1/2} u^{I,0}(0,t) v_{x}^{B,1} - \varepsilon^{1/2} u_{x}^{B,1} v^{I,0}(0,t) - \varepsilon^{1/2} (u^{B,1} v^{B,0})_x ,
\]

and
\[
\varepsilon u_{I}^{b,1} = - u^{I,0}(1,t) v_{x}^{b,0} ,
\]
\[
\varepsilon u_{I}^{b,2} = - (x-1) u^{I,0}(1,t) v_{x}^{b,0} - \varepsilon^{1/2} u^{I,1}(1,t) v_{x}^{b,0} - u^{I,0}(1,t) v_{x}^{b,0}
\]
\[
- \varepsilon^{1/2} u^{I,0}(1,t) v_{x}^{b,1} - \varepsilon^{1/2} u_{x}^{b,1} v^{I,0}(1,t) - \varepsilon^{1/2} (u^{b,1} v^{b,0})_x .
\]
Then feeding (4.9) on the above four expressions and rearranging the results, we have
\[ f^\varepsilon = [(u^{I,0}(x, t) - u^{I,0}(0, t) - x v_x^{I,0}(0, t)) v_x^{B,0}] 
+ [(u^{I,0}(x, t) - u^{I,0}(1, t) - (x - 1) u_x^{I,0}(1, t)) v_x^{b,0}] 
+ [(u_x^{I,0}(x, t) - u_x^{I,0}(0, t)) v_{B,0} + (u_x^{I,0}(x, t) - u_x^{I,0}(1, t)) v_{b,0}] 
+ \varepsilon^{1/2}[(u^{I,0}(x, t) - u^{I,0}(0, t)) v_x^{B,1} + (u^{I,1}(x, t) - u^{I,1}(0, t)) v_x^{B,0} + u_x^{B,1}(v^{I,0}(x, t) - v^{I,0}(0, t))] 
+ \varepsilon^{1/2}[(u^{I,0}(x, t) - u^{I,0}(1, t)) v_x^{b,1} + (u^{I,1}(x, t) - u^{I,1}(1, t)) v_x^{b,0} + u_x^{b,1}(v^{I,0}(x, t) - v^{I,0}(1, t))]
\]
+ \varepsilon^{1/2}[u_x^{B,0}(v^{B,1} + v^{b,1}) + u_x^{b,1}(v^{B,0} + v^{b,0}) + (u_x^{B,1} + u_x^{b,1}) v_x^{I,0}] 
+ \varepsilon^{1/2}[u_x^{B,1} v^{b,0} + u_x^{b,1} v^{B,0} + u_x^{b,1} v^{b,1} + u_x^{1} v^{B,0}] 
+ \varepsilon[(u_x^{I,1} + u_x^{B,1} + u_x^{b,1})(v_x^{I,1} + v_x^{b,1})] 
+ \varepsilon[(u_x^{B,2} + u_x^{b,2})(v_x^{B,0} + v_x^{b,0}) + \varepsilon^{1/2} v_x^{I,1} + \varepsilon^{1/2} v_x^{B,1} + \varepsilon^{1/2} v_x^{b,1}] 
- [\varepsilon^{1/2} u_x^{B,1} + \varepsilon^{1/2} u_x^{b,1} + \varepsilon u_x^{2} + \varepsilon u_x^{b,2}] + F^\varepsilon 
\]
\[ := \sum_{i=1}^{10} K_i + F^\varepsilon . \]

(4.11)

We proceed to estimate \( K_i \) (\( 1 \leq i \leq 10 \)). Recalling that \( x = \varepsilon^{1/2} z \), then by Taylor’s formula, (4.5) and Lemma 3.1-Lemma 3.2, \( K_1 \) is estimated as follows:

\[
\|K_1\|_{L^2(0,T;L^2)} = \varepsilon \left\| \frac{u^{I,0}(x, t) - u^{I,0}(0, t) - x u_x^{I,0}(0, t)}{x^2} \cdot z^2 v_x^{B,0} \right\|_{L^2(0,T;L^2)} 
\leq \varepsilon \|u_{xx}^{I,0}\|_{L^2(0,T;L^\infty)} \|z^2 v_x^{B,0}\|_{L^\infty(0,T;L^2)} 
\leq C_0 \varepsilon^{3/4} \|u^{I,0}\|_{L^2(0,T;H^3)} \|z^2 v_x^{B,0}\|_{L^\infty(0,T;L^2)} 
\leq C \varepsilon^{3/4}.
\]

Similarly, by using (4.6) we have
\[
\|K_2\|_{L^2(0,T;L^2)} \leq \varepsilon \left\| \frac{u^{I,0}(x, t) - u^{I,0}(1, t) - (x - 1) u_x^{I,0}(1, t)}{(x - 1)^2} \cdot \xi^2 v_x^{b,0} \right\|_{L^2(0,T;L^2)} \leq C \varepsilon^{3/4}.
\]

Similar arguments further give
\[
\|K_i\|_{L^2(0,T;L^2)} \leq C \varepsilon^{3/4}, \quad i = 3, 4, 5.
\]

By the Sobolev embedding inequality, (4.5)-(4.6) and Lemma 3.1-Lemma 3.6 we obtain
\[
\|K_6\|_{L^2(0,T;L^2)} \leq C_0 \varepsilon^{3/4} \|u^{I,0}\|_{L^\infty(0,T;H^2)} (\|v_x^{B,1}\|_{L^2(0,T;L^2)} + \|v_x^{b,1}\|_{L^2(0,T;L^2)}) 
+ C_0 \varepsilon^{3/4} \|u_x^{I,1}\|_{L^\infty(0,T;H^2)} (\|v_x^{B,0}\|_{L^2(0,T;L^2)} + \|v_x^{b,0}\|_{L^2(0,T;L^2)}) 
+ C_0 \varepsilon^{3/4} \|v_x^{I,0}\|_{L^\infty(0,T;H^2)} (\|u_x^{B,1}\|_{L^2(0,T;L^2)} + \|u_x^{b,1}\|_{L^2(0,T;L^2)}) 
\leq C \varepsilon^{3/4}.
\]

Then using a similar argument as estimating \( K_6 \) and recalling \( 0 < \varepsilon < 1 \), one infers that
\[
\|K_i\|_{L^2(0,T;L^2)} \leq C \varepsilon^{3/4}, \quad 8 \leq i \leq 10.
\]

We proceed to bound each term in \( K_7 \). Indeed for \( 0 < x < 1/2 \), it follows that \( -\infty < \xi = \frac{x-1}{\sqrt{\varepsilon}} < -\frac{1}{2\sqrt{\varepsilon}} \). Thus, by transformation (2.2) and the Sobolev embedding inequality, one deduces for
fixed $t \in [0, T]$ and $m \in \mathbb{N}_+$ that

$$
\int_0^1 \left( u_{x}^{B,1} v^{b,0} \right)^2 dx \\
= \int_0^1 \left[ \frac{u_{x}^{B,1}}{\sqrt{\varepsilon}(x, t)} v^{b,0} \left( \frac{x - 1}{\sqrt{\varepsilon}(x, t)} \right) \right]^2 dx \\
\leq \int_0^1 \left[ \frac{u_{x}^{B,1}}{\sqrt{\varepsilon}(x, t)} \right]^2 dx \cdot (2\sqrt{\varepsilon})^{2m} \left\| \left( -\frac{1}{2\sqrt{\varepsilon}(x, t)} \right)^m v^{b,0}(\xi, t) \right\|^2_{L^\infty_T(\infty, -\frac{1}{2\sqrt{\varepsilon}})} \\
\leq \varepsilon^{-1/2} \int_0^1 \left[ \frac{u_{x}^{B,1}}{\sqrt{\varepsilon}}(z, t) \right]^2 dz \cdot (2\sqrt{\varepsilon})^{2m} \left\| \left( \xi^m \right)^{b,0}(\xi, t) \right\|^2_{L^\infty_T(-\infty, 0)} \\
\leq C_0 \varepsilon^{(2m-1)/2} \left\| u_{x}^{B,1}(z, t) \right\|^2_{L^2_T} \left\| \left( \xi^m \right)^{b,0}(\xi, t) \right\|^2_{H^1_T} \\
\leq C \varepsilon^{1/2},
$$

where Lemma 3.2 and Lemma 3.3 have been used. Similarly, for $\frac{1}{2} < x < 1$ one has that

$$
\frac{1}{2\sqrt{\varepsilon}} < z = \frac{x}{\sqrt{\varepsilon}} < \infty 
$$

and for $m \in \mathbb{N}_+$ that

$$
\int_{\frac{1}{2}}^1 \left( u_{x}^{B,1} v^{b,0} \right)^2 dx \\
\leq \varepsilon^{1/2} \int_{-\infty}^0 \left\| v^{b,0}(\xi, t) \right\|^2 d\xi \cdot \varepsilon^{-1} \left\| u_{x}^{B,1}(z, t) \right\|^2_{L^\infty_T(\frac{1}{\sqrt{\varepsilon}}, \infty)} \\
\leq \varepsilon^{-1/2} \left\| v^{b,0}(\xi, t) \right\|^2_{L^2_T} \cdot \varepsilon^{-1} (2\sqrt{\varepsilon})^{2m} \left\| z^m u_{x}^{B,1}(z, t) \right\|^2_{L^\infty_T(0, \infty)} \\
\leq C \varepsilon^{1/2},
$$

Combining the above two estimates, we end up with $\| u_{x}^{B,1} v^{b,0} \|_{L^2(0,T;L^2)} \leq C \varepsilon^{1/4}$. By similar arguments, one derives that

$$
\| u_x^{B,1} v_x^{b,0} \|_{L^2(0,T;L^2)} + \| u_x^{b,1} v_x^{B,0} \|_{L^2(0,T;L^2)} + \| u_x^{b,1} v_x^{B,0} \|_{L^2(0,T;L^2)} \leq C \varepsilon^{1/4}.
$$

Thus $\| K_7 \|_{L^2(0,T;L^2)} \leq C \varepsilon^{3/4}$. 

For the last term $F_\varepsilon$, we first note for any integer $m \geq 2$ that

$$
\| u^{b,1}(-\varepsilon^{-1/2}, t) \|_{L^\infty(0,T)} = \varepsilon^{m/2} \| \left( -\varepsilon^{-1/2} \right)^m u^{b,1}(-\varepsilon^{-1/2}, t) \|_{L^\infty(0,T)} \\
\leq \varepsilon^{m/2} \| \left( \xi^m \right)^{b,1}(\xi, t) \|_{L^\infty(0,T;L^\infty)} \\
\leq C_0 \varepsilon^{m/2} \| \left( \xi^m \right)^{b,1}(\xi, t) \|_{L^\infty(0,T;H^1)} \\
\leq C \varepsilon^{m/2}
$$

and that $\| u^{B,2}(0,t) \|_{L^\infty(0,T)} \leq C_0 \| u^{B,2} \|_{L^\infty(0,T;H^1)} \leq C$. By similar arguments, we can estimate other terms in $b_1^2$, $b_2^2$ and conclude that

$$
\| b_1^2 \|_{L^\infty(0,T;H^1)} + \| b_2^2 \|_{L^\infty(0,T;H^1)} \leq C(\varepsilon^{m/2} + \varepsilon) \leq C \varepsilon. \quad (4.12)
$$

Similar arguments further entail that

$$
\| b_1^2 \|_{L^2(0,T;H^1)} + \| b_2^2 \|_{L^2(0,T;H^1)} \leq C(\varepsilon^{m/2} + \varepsilon) \leq C \varepsilon. \quad (4.13)
$$
Then substituting (4.12)-(4.13) into the definition of $F^\varepsilon$ in (4.10) and using $0 < \varepsilon < 1$ and (4.5)-(4.6), one has
\[
\| F^\varepsilon \|_{L^2(0,T;L^2)} \leq C_0 \| b_1^\varepsilon \|_{L^\infty(0,T;H^1)} \left\{ \| v^{I,0} \|_{L^2(0,T;H^1)} + \varepsilon^{-1/4} \| v^{B,0} \|_{L^2(0,T;H^2)} + \varepsilon^{-1/4} \| v^{b,0} \|_{L^2(0,T;H^2)} \right\} \\
+ \varepsilon^{1/2} \| v^{I,1} \|_{L^2(0,T;H^1)} + \varepsilon^{1/4} \| v^{B,1} \|_{L^2(0,T;H^2)} + \varepsilon^{1/4} \| v^{b,1} \|_{L^2(0,T;H^2)} \\
+ C_0 \| b_2^\varepsilon \|_{L^\infty(0,T;H^1)} \left\{ \| u^{I,0} \|_{L^2(0,T;H^1)} + \varepsilon^{1/2} \| u^{I,1} \|_{L^2(0,T;H^1)} + \varepsilon^{1/4} \| u^{B,1} \|_{L^2(0,T;H^2)} \right\} \\
+ \varepsilon^{1/4} \| u^{b,1} \|_{L^2(0,T;H^1)} + \varepsilon^{3/4} \| u^{B,2} \|_{L^2(0,T;H^2)} + \varepsilon^{3/4} \| u^{b,2} \|_{L^2(0,T;H^2)} \\
+ T^{1/2} \| b_1^\varepsilon \|_{L^\infty(0,T;H^1)} \| b_2^\varepsilon \|_{L^\infty(0,T;L^2)} + \| b_3^\varepsilon \|_{L^2(0,T;L^2)} \\
\leq C \varepsilon^{3/4}.
\]
Collecting the above estimates for $K_i$ ($1 \leq i \leq 10$) and $F^\varepsilon$, from (4.11) one derives (4.8) and finishes the proof.

**Lemma 4.2.** Let $0 < T < \infty$, $0 < \varepsilon < 1$ and $g^\varepsilon$ be as defined in (4.4). Then
\[
\| g^\varepsilon \|_{L^2(0,T;L^2)} \leq C \varepsilon^{3/4}.
\]

**Proof.** Substituting the definition for $\tilde{U}^\varepsilon$ and $V^\varepsilon$ into $g^\varepsilon$ in (4.4), then using the second equations in (2.3), (2.8), (6.16) and (6.17) (see Appendix), we have
\[
g^\varepsilon = [\varepsilon v^{I,0} + \varepsilon^{3/2} v^{I,1}] + [\varepsilon^{3/2} (v^{B,1} + v^{b,1}) + \varepsilon (u^{B,2} + u^{b,2}) - \varepsilon^{1/2} (v^{B,1} + v^{b,1})] \\
- [2\varepsilon \tilde{V}^\varepsilon \tilde{V}_x^\varepsilon] + [b_1^\varepsilon - b_2^\varepsilon]
\]
\begin{equation}
:= \sum_{i=11}^{14} K_i.
\end{equation}

We next estimate $K_i$ ($11 \leq i \leq 14$). Lemma 3.1 and Lemma 3.4 imply that $\| K_{11} \|_{L^2(0,T;L^2)} \leq C \varepsilon$. Using (2.2), (4.5)-(4.6) and Lemma 3.5-Lemma 3.6, we estimate $K_{12}$ as follows:
\[
\| K_{12} \|_{L^2(0,T;L^2)} \leq \varepsilon^{3/4} \left\{ \| v^{b,1}_x \|_{L^2(0,T;L^2)} + \| v^{b,1}_\xi \|_{L^2(0,T;L^2)} + \| u^{b,2}_x \|_{L^2(0,T;L^2)} \\
+ \| u^{b,2}_\xi \|_{L^2(0,T;L^2)} + \| v^{b,1}_t \|_{L^2(0,T;L^2)} + \| v^{b,1}_t \|_{L^2(0,T;L^2)} \right\} \leq C \varepsilon^{3/4}.
\]
To bound $K_{13}$, we first estimate $\| \tilde{V}^\varepsilon \|_{L^\infty(0,1) \times (0,T)}$ and $\| \tilde{V}_x^\varepsilon \|_{L^\infty(0,T;L^2)}$. For any $G_1(z,t) \in L^p(0,T;H^1_z)$, $G_2(\xi,t) \in L^p(0,T;L^2_\xi)$ with $1 \leq p \leq \infty$, it follows from the Sobolev embedding inequality that
\begin{equation}
\begin{aligned}
\| G_1 \left( \frac{x}{\sqrt{\varepsilon}}, t \right) \|_{L^p(0,T;L^\infty)} &\leq \| G_1(z,t) \|_{L^p(0,T;L^\infty)} \leq C_0 \| G_1 \|_{L^p(0,T;H^1)} \leq C, \\
\| G_2 \left( \frac{x-1}{\sqrt{\varepsilon}}, t \right) \|_{L^p(0,T;L^\infty)} &\leq \| G_2(\xi,t) \|_{L^p(0,T;L^\infty)} \leq C_0 \| G_2 \|_{L^p(0,T;H^1)} \leq C.
\end{aligned}
\end{equation}

Then by the definition of $\tilde{V}^\varepsilon$, (4.15), Lemma 3.1-Lemma 3.6 and (4.13), we deduce that
\[
\| \tilde{V}^\varepsilon \|_{L^\infty(0,1) \times (0,T)} \leq \| v^{I,0} \|_{L^\infty(0,1) \times (0,T)} + \| v^{B,0} \|_{L^\infty(0,T;L^\infty)} + \| v^{b,0} \|_{L^\infty(0,T;L^\infty)} + C_0 \| b_2^\varepsilon \|_{L^\infty(0,T;H^1)} \\
+ \varepsilon^{1/2} \| v^{I,1} \|_{L^\infty(0,1) \times (0,T)} + \| v^{B,1} \|_{L^\infty(0,T;L^\infty)} + \| v^{b,1} \|_{L^\infty(0,T;L^\infty)} \\
\leq C(1 + \varepsilon^{1/2} + \varepsilon^{m/2}) \leq C,
\]

where the assumption $0 < \varepsilon < 1$ has been used. Moreover (4.5), (4.6) and (4.13) lead to
\[
\|\tilde{V}_x^\varepsilon\|_{L^\infty(0,T;L^2)} 
\leq \|v_x^0\|_{L^\infty(0,T;L^2)} + \varepsilon^{-1/4}(\|v_x^{B_0}\|_{L^\infty(0,T;L^2)} + \|v_x^{b_0}\|_{L^\infty(0,T;L^2)}) + C\varepsilon^{m/2} 
+ \varepsilon^{1/2}(\|v_x^{B_1}\|_{L^\infty(0,T;L^2)} + \|v_x^{b_1}\|_{L^\infty(0,T;L^2)})
\]
\[
\leq C\varepsilon^{-1/4}.
\]

Thus the above two estimates indicate that
\[
\|K_{15}\|_{L^2(0,T;L^2)} \leq C\varepsilon\|\tilde{V}_x^\varepsilon\|_{L^\infty((0,1] \times [0,T])}\|\tilde{V}_x^\varepsilon\|_{L^\infty(0,T;L^2)} \leq C\varepsilon^{3/4}.
\]
Finally, the estimate for $K_{14}$ follows from (4.12), (4.13) and the assumption $0 < \varepsilon < 1$ that
\[
\|K_{14}\|_{L^2(0,T;L^2)} \leq \|b_2^*\|_{L^2(0,T;H^1)} + \|b_2^*\|_{L^2(0,T;L^2)} \leq C\varepsilon \leq C\varepsilon^{3/4}.
\]
Then inserting the above estimates for $K_i$ ($11 \leq i \leq 14$) into (4.14) yields the desired estimate for $g^\varepsilon$.

Next lemma gives the estimate for $U^\varepsilon$, $V^\varepsilon$ in $L^\infty(0,T;L^2)$.

**Lemma 4.3.** Let $0 < T < \infty$ and $0 < \varepsilon < 1$. Then there exists a constant $C$, independent of $\varepsilon$, such that
\[
\|U^\varepsilon\|_{L^\infty(0,T;L^2)} + \|V^\varepsilon\|_{L^\infty(0,T;L^2)} + \|U_x^\varepsilon\|_{L^2(0,T;L^2)} + \|V_x^\varepsilon\|_{L^2(0,T;L^2)} \leq C\varepsilon^{1/2}.
\]

**Proof.** Taking the $L^2$ inner product of the first equation of (4.3) with $2U_x^\varepsilon$, then using integration by parts to have
\[
\frac{d}{dt}\|U_x^\varepsilon(t)\|_{L^2}^2 + 2\|U_x^\varepsilon(t)\|_{L^2}^2 = -2\varepsilon^{1/2}\int_0^1 U^\varepsilon V_x^\varepsilon U_x^\varepsilon dx - 2\int_0^1 (U^\varepsilon \tilde{V}_x^\varepsilon + V_x^\varepsilon U_x^\varepsilon U_x^\varepsilon dx
\]
\[
+ 2\varepsilon^{-1/2}\int_0^1 f_x^\varepsilon U_x^\varepsilon dx
\]
\[
:= M_1(t) + M_2(t) + M_3(t).
\]
We next estimate $M_i(t)$ ($i = 1, 2, 3$). First, (4.7) gives
\[
M_1(t) \leq C_0 \varepsilon^{1/2}\|U^\varepsilon(t)\|_{L^2}^{1/2}\|U_x^\varepsilon(t)\|_{L^2}^{3/2}\|V^\varepsilon(t)\|_{L^2}
\]
\[
\leq \frac{1}{4}\|U_x^\varepsilon(t)\|_{L^2}^2 + C_0 \varepsilon \|V^\varepsilon(t)\|_{L^2}^2 + \|U_x^\varepsilon(t)\|_{L^2}^2.
\]
For the term $\|V^\varepsilon(t)\|_{L^2}^2$, we use the definition of $V^\varepsilon$, Lemma 2.1 and (4.16) to get
\[
\|\tilde{V}_x^\varepsilon\|_{L^\infty(0,T;L^\infty)} \leq \varepsilon^{-1/2}((\|v^\varepsilon\|_{L^\infty(0,T;L^\infty)} + \|\tilde{V}_x^\varepsilon\|_{L^\infty(0,T;L^2)}) \leq C\varepsilon^{-1/2},
\]
which, substituted into the above estimate for $M_1(t)$ gives rise to $M_1(t) \leq \frac{1}{4}\|U_x^\varepsilon(t)\|_{L^2}^2 + C \|V^\varepsilon(t)\|_{L^2}^2$. By a similar argument as deriving (4.16), one infers that
\[
\|\tilde{V}_x^\varepsilon\|_{L^\infty((0,1] \times [0,T])} \leq C,
\]
which along with (4.16) leads to
\[
M_2(t) \leq \frac{1}{4}\|U_x^\varepsilon(t)\|_{L^2}^2 + 8\|U_x^\varepsilon(t)\|_{L^2}^2\|\tilde{V}_x^\varepsilon(t)\|_{L^\infty}^2 + 8\|V^\varepsilon(t)\|_{L^2}^2\|\tilde{V}_x^\varepsilon(t)\|_{L^\infty}^2
\]
\[
\leq \frac{1}{4}\|U_x^\varepsilon(t)\|_{L^2}^2 + C(\|U^\varepsilon(t)\|_{L^2}^2 + \|V^\varepsilon(t)\|_{L^2}^2).
\]
For the last term $M_3(t)$, we have by the Cauchy-Schwarz inequality that $M_3(t) \leq \|U^\varepsilon(t)\|_{L^2}^2 + \varepsilon^{-1}\|f^\varepsilon(t)\|_{L^2}^2$. Substituting the above estimates of $M_i(t)$ ($1 \leq i \leq 3$) into (4.18), we arrive at
\[
\frac{d}{dt}\|U^\varepsilon(t)\|_{L^2}^2 + \frac{3}{2}\|U_x^\varepsilon(t)\|_{L^2}^2 \leq C(\|U^\varepsilon(t)\|_{L^2}^2 + \|V^\varepsilon(t)\|_{L^2}^2) + \varepsilon^{-1}\|f^\varepsilon(t)\|_{L^2}^2.
\]
We turn to estimate \( V^\varepsilon \). Multiplying the second equation of (4.3) by \( 2V^\varepsilon \) in \( L^2 \) and using the integration by parts to derive

\[
\frac{d}{dt}\|V^\varepsilon(t)\|_{L^2}^2 + 2\varepsilon\|V_{xx}^\varepsilon(t)\|_{L^2}^2 = -4\varepsilon^{3/2}\int_0^1 V^\varepsilon V^\varepsilon V_x^\varepsilon dx + 4\varepsilon\int_0^1 V^\varepsilon \tilde{V} \dot{V}_x^\varepsilon dx + 2\int_0^1 U_x^\varepsilon V^\varepsilon dx + 2\varepsilon^{-1/2}\int_0^1 g V^\varepsilon dx
\]

(4.22)

We proceed to bound \( M_i(t) \) (4 \( \leq i \leq 7 \)). Applying (4.7) to \( V^\varepsilon \) together with (4.19) leads to

\[
M_4(t) \leq C_0\varepsilon^{3/2}\|V_x^\varepsilon(t)\|_{L^2}^{3/2}\|V^\varepsilon(t)\|_{L^2}^{3/2}
\]

\[
\leq \frac{1}{4}\varepsilon\|V_x^\varepsilon(t)\|_{L^2}^2 + \frac{1}{4}\varepsilon\|V^\varepsilon(t)\|_{L^2}^2 + C\varepsilon\|V^\varepsilon(t)\|_{L^2}^2.
\]

We employ the Cauchy-Schwarz inequality and (4.16) to deduce that

\[
M_5(t) \leq \frac{1}{4}\varepsilon\|V_x^\varepsilon(t)\|_{L^2}^2 + 16\varepsilon\|\tilde{V}^\varepsilon(t)\|_{L^\infty}\|V^\varepsilon(t)\|_{L^2}^2 \leq \frac{1}{4}\varepsilon\|V_x^\varepsilon(t)\|_{L^2}^2 + C\varepsilon\|V^\varepsilon(t)\|_{L^2}^2.
\]

Finally, the estimates for \( M_6(t) \) and \( M_7(t) \) follow from the Cauchy-Schwarz inequality that

\[
M_6(t) \leq \frac{1}{4}\|U_{xx}^\varepsilon(t)\|_{L^2}^2 + 4\|V^\varepsilon(t)\|_{L^2}^2, \quad M_7(t) \leq \|V^\varepsilon(t)\|_{L^2}^2 + \varepsilon^{-1}\|g^\varepsilon(t)\|_{L^2}^2.
\]

We turn to estimate \( V^\varepsilon \). Multiplying the second equation of (4.3) by \( -2\varepsilon V_{xx}^\varepsilon \), and using integration by parts, we obtain

\[
\frac{d}{dt}(\varepsilon\|V_x^\varepsilon(t)\|_{L^2}^2) + 2\varepsilon\|V_{xx}^\varepsilon(t)\|_{L^2}^2 = 4\varepsilon^{3/2}\int_0^1 V_x^\varepsilon V_x^\varepsilon V_x^\varepsilon dx + 4\varepsilon^2\int_0^1 (V_x^\varepsilon \tilde{V}^\varepsilon)_x V_x^\varepsilon dx
\]

\[- 2\varepsilon\int_0^1 U_{xx}^\varepsilon V_x^\varepsilon dx - 2\varepsilon^{1/2}\int_0^1 g V_{xx}^\varepsilon dx
\]

(4.23)

We proceed to estimate \( R_i(t) \) for 1 \( \leq i \leq 4 \). By (4.7) we deduce that

\[
R_i(t) \leq C_0\varepsilon^{5/2}\|V_x^\varepsilon(t)\|_{L^2}^2\|V_{xx}^\varepsilon(t)\|_{L^2} \leq \frac{1}{4}\varepsilon^2\|V_{xx}^\varepsilon(t)\|_{L^2}^2 + C_0\varepsilon^2\|V_x^\varepsilon(t)\|_{L^2}^4.
\]
Similarly, it follows from (4.7), (4.16) and (4.17) that
\[
R_2(t) \leq C_0 \varepsilon^2 \|V^\varepsilon_x(t)\|_{L^2} \|\tilde{V}^\varepsilon_x(t)\|_{H^1} \|V^\varepsilon_{xx}(t)\|_{L^2} \\
\leq \frac{1}{4} \varepsilon^2 \|V^\varepsilon_{xx}(t)\|^2_{L^2} + C_0 \varepsilon^2 \|\tilde{V}^\varepsilon_x(t)\|^2_{L^2} + \|\tilde{V}^\varepsilon_x(t)\|^2_{L^2} \|V^\varepsilon_x(t)\|^2_{L^2} \\
\leq \frac{1}{4} \varepsilon^2 \|V^\varepsilon_{xx}(t)\|^2_{L^2} + C(\varepsilon^2 + \varepsilon^{3/2}) \|V^\varepsilon_x(t)\|^2_{L^2}.
\]

For $R_3(t)$ and $R_4(t)$, we employ the Cauchy-Schwarz inequality to have
\[
R_3(t) \leq \frac{1}{4} \varepsilon^2 \|V^\varepsilon_{xx}(t)\|^2_{L^2} + 4 \|U^\varepsilon_x(t)\|^2_{L^2} \quad \text{and} \quad R_4(t) \leq \frac{1}{4} \varepsilon^2 \|V^\varepsilon_{xx}(t)\|^2_{L^2} + 4 \varepsilon^{-1} \|g^\varepsilon(t)\|^2_{L^2}.
\]
Collecting the above estimates of $R_i(t)$ ($1 \leq i \leq 4$) and using (4.23), we end up with
\[
\frac{d}{dt}(\varepsilon \|V^\varepsilon_x(t)\|^2_{L^2}) + \varepsilon^2 \|V^\varepsilon_{xx}(t)\|^2_{L^2} \\
\leq C(\varepsilon^2 \|V^\varepsilon_x(t)\|^2_{L^2} + \varepsilon + \varepsilon^{1/2}(\varepsilon \|V^\varepsilon_x(t)\|^2_{L^2}) + 4 \|U^\varepsilon_x(t)\|^2_{L^2} + \varepsilon^{-1} \|g^\varepsilon(t)\|^2_{L^2}),
\]
which, along with Gronwall’s inequality, Lemma 4.2-Lemma 4.3 and $0 < \varepsilon < 1$ yields
\[
\varepsilon \|V^\varepsilon_x\|^2_{L^2(0,T;L^2)} + \varepsilon^2 \|V^\varepsilon_{xx}\|^2_{L^2(0,T;L^2)} \leq C\varepsilon^{1/2}. \tag{4.24}
\]

We turn to estimate $U^\varepsilon_x$. Taking the $L^2$ inner product of the first equation of (4.3) against $-2U^\varepsilon_{xx}$ and using integration by parts to get
\[
\frac{d}{dt} \|U^\varepsilon_x(t)\|^2_{L^2} + 2 \|U^\varepsilon_{xx}(t)\|^2_{L^2} = -2\varepsilon^{1/2} \int_0^1 (U^\varepsilon_x V^\varepsilon)_{,xx} U^\varepsilon_{xx} \, dx - 2 \int_0^1 (U^\varepsilon_x \tilde{V}^\varepsilon)_{,xx} U^\varepsilon_{xx} \, dx \\
- 2 \int_0^1 (V^\varepsilon \tilde{U}^\varepsilon)_{,xx} U^\varepsilon_{xx} \, dx - 2\varepsilon^{-1/2} \int_0^1 f^\varepsilon U^\varepsilon_{xx} \, dx \tag{4.25}
\]
By (4.7) and (4.24), we estimate $R_5(t)$ as
\[
R_5(t) \leq \frac{1}{4} \|U^\varepsilon_{xx}(t)\|^2_{L^2} + C_0 \varepsilon \|V^\varepsilon_x(t)\|^2_{L^2} \|U^\varepsilon_x(t)\|^2_{L^2} \leq \frac{1}{4} \|U^\varepsilon_{xx}(t)\|^2_{L^2} + C \varepsilon^{1/2} \|U^\varepsilon_x(t)\|^2_{L^2}.
\]
Similarly, we estimate $R_6(t)$ from (4.7), (4.16) and (4.17) as
\[
R_6(t) \leq \frac{1}{4} \|U^\varepsilon_{xx}(t)\|^2_{L^2} + C_0 \|\tilde{V}^\varepsilon_x(t)\|^2_{L^2} \|U^\varepsilon_x(t)\|^2_{L^2} + C \|\tilde{V}^\varepsilon_x(t)\|^2_{L^2} \|U^\varepsilon_x(t)\|^2_{L^2} \\
\leq \frac{1}{4} \|U^\varepsilon_{xx}(t)\|^2_{L^2} + C(1 + \varepsilon^{-1/2}) \|U^\varepsilon_x(t)\|^2_{L^2}.
\]
To bound $R_7(t)$, we use the definition of $\tilde{U}^\varepsilon$ and a similar argument as deriving (4.17) to get
\[
\|\tilde{U}^\varepsilon_x\|^2_{L^2(0,T;L^2)} \leq C(1 + \varepsilon^{1/2} + \varepsilon^{1/4} + \varepsilon^{3/4} + \varepsilon) \leq C,
\]
where $0 < \varepsilon < 1$ has been used. The above estimate in conjunction with (4.20) and (4.24) gives
\[
R_7(t) \leq \frac{1}{4} \|U^\varepsilon_{xx}(t)\|^2_{L^2} + C_0(\|\tilde{U}^\varepsilon(t)\|^2_{L^2} + \|\tilde{V}^\varepsilon_x(t)\|^2_{L^2}) \|V^\varepsilon_x(t)\|^2_{L^2} \leq \frac{1}{4} \|U^\varepsilon_x(t)\|^2_{L^2} + C \varepsilon^{-1/2}.
\]
Lastly, the Cauchy-Schwarz inequality yields $R_8(t) \leq \frac{1}{4} \|U^\varepsilon_{xx}(t)\|^2_{L^2} + 4 \varepsilon^{-1} \|f^\varepsilon(t)\|^2_{L^2}$. Feeding (4.25) on the above estimates of $R_i(t)$ ($5 \leq i \leq 8$) leads to
\[
\frac{d}{dt} \|U^\varepsilon_x(t)\|^2_{L^2} + \|U^\varepsilon_{xx}(t)\|^2_{L^2} \leq C(\varepsilon^{-1/2} + 1 + \varepsilon^{1/2}) \|U^\varepsilon_x(t)\|^2_{L^2} + C \varepsilon^{-1/2} + 4 \varepsilon^{-1} \|f^\varepsilon(t)\|^2_{L^2},
\]
which, upon integration over $(0,t)$ with $t \leq T$ gives rise to
\[
\|U^\varepsilon_x\|^2_{L^2(0,T;L^2)} + \|U^\varepsilon_{xx}\|^2_{L^2(0,T;L^2)} \leq C \varepsilon^{-1/2},
\]
where Lemma 4.1, Lemma 4.3 and $0 < \varepsilon < 1$ have been used. The above estimate along with (4.24) completes the proof.

\qed
4.3. Proof of Theorem 2.1. To prove Theorem 2.1, it suffices to estimate \( \| R_1^\varepsilon \|_{L^\infty([0,1] \times [0,T])} \) and \( \| R_2^\varepsilon \|_{L^\infty([0,1] \times [0,T])} \). For this, we first estimate \( U^\varepsilon \) and \( V^\varepsilon \) in \( L^\infty([0,1] \times [0,T]) \) by (4.7), Lemma 4.3-Lemma 4.4 and get

\[
\| U^\varepsilon \|_{L^\infty([0,1] \times [0,T])} \leq C_0 \| U^\varepsilon \|_{L^2(0,T;L^2)}^{1/2} \| U^\varepsilon \|_{L^\infty([0,T];L^2)}^{1/2} \leq C, \tag{4.26}
\]

Then the estimate for \( R_1^\varepsilon \) follows from (4.15), Lemma 3.2-Lemma 3.6, (4.12) and (4.26) that

\[
\| R_1^\varepsilon \|_{L^\infty([0,1] \times [0,T])} \leq C_0 \varepsilon^{1/2} \left( \| u^{1,1} \|_{L^\infty(0,T;H^1)} + \| u^{B,1} \|_{L^\infty(0,T;H^1)} + \| b^{h,1} \|_{L^\infty(0,T;H^1)} \right)
+ C_0 \varepsilon \left( \| u^{B,2} \|_{L^\infty(0,T;H^1)} + \| b^{h,2} \|_{L^\infty(0,T;H^1)} \right) + C_0 \| b^c \|_{L^\infty(0,T;H^1)} + \varepsilon^{1/2} \| U^\varepsilon \|_{L^\infty([0,1] \times [0,T])}
\leq C \varepsilon^{1/2},
\]

where \( 0 < \varepsilon < 1 \) has been used. Similarly, by (4.15), Lemma 3.4-Lemma 3.6, (4.13), (4.26) and \( 0 < \varepsilon < 1 \), we have

\[
\| R_2^\varepsilon \|_{L^\infty([0,1] \times [0,T])} \leq C_0 \varepsilon^{1/2} \left( \| u^{1,1} \|_{L^\infty(0,T;H^1)} + \| u^{B,1} \|_{L^\infty(0,T;H^1)} + \| b^{h,1} \|_{L^\infty(0,T;H^1)} \right)
+ C_0 \varepsilon \left( \| u^{B,2} \|_{L^\infty(0,T;H^1)} + \| b^{h,2} \|_{L^\infty(0,T;H^1)} \right) + C_0 \| b^c \|_{L^\infty(0,T;H^1)} + \varepsilon^{1/2} \| V^\varepsilon \|_{L^\infty([0,1] \times [0,T])}
\leq C \varepsilon^{1/2}.
\]

The above two estimates along with (4.1) imply (2.13) and complete the proof of Theorem 2.1.

\[ \square \]

5. Proof of Theorem 2.2

We are now in a position to prove Theorem 2.2 by converting the result of Theorem 2.1 to the pre-transformed chemotaxis model (2.15).

Proof of Theorem 2.2. Let \( (u^\varepsilon, c^\varepsilon) \) and \( (u^0, c^0) \) be solutions of (2.15) with \( \varepsilon > 0 \) and \( \varepsilon = 0 \), respectively. The convergence rate in (2.17) between \( u^\varepsilon \) and \( u^0 \) is a direct consequence of Theorem 2.1. We are left to prove the convergence for \( c^\varepsilon \) in (2.17) and for \( c^2_\varepsilon \) in (2.18). Indeed from the second equation of (2.15) one deduces that

\[
\begin{align*}
\frac{\partial c^\varepsilon(x,t)}{\partial t} &= c^\varepsilon(x,t) \left[ \frac{\partial}{\partial x} \left( \ln c^\varepsilon(x,t) \right) \right] + \frac{\partial}{\partial x} \left( \ln c^0(x,t) \right) \\
\frac{\partial c^0(x,t)}{\partial t} &= -u^0 \varepsilon \\
\end{align*}
\]

where \( v^\varepsilon = -\frac{\partial}{\partial x} \left( \ln c^\varepsilon(x,t) \right) \). We consider the difference of the two equations:

\[
\left( \ln c^\varepsilon - \ln c^0 \right) = \varepsilon \left( v^\varepsilon \right)^2 - \varepsilon v^\varepsilon_x - u^\varepsilon - u^0,
\]

which, upon integration with respect to \( t \), gives rise to

\[
\frac{c^\varepsilon(x,t)}{c^0(x,t)} = \frac{c^\varepsilon(x,0)}{c^0(x,0)} \exp \left\{ \int_0^t \left[ (u^\varepsilon - u^0) + \varepsilon (v^\varepsilon)^2 - \varepsilon v^\varepsilon_x \right] \, d\tau \right\}.
\]

It follows from the initial condition \( c^\varepsilon(x,0) = c^0(x,0) = c_0(x) \) that

\[
|c^\varepsilon(x,t) - c^0(x,t)| = |c^0(x,t)| \left| \exp \left\{ \int_0^t \left[ -(u^\varepsilon - u^0) + \varepsilon (v^\varepsilon)^2 - \varepsilon v^\varepsilon_x \right] \, d\tau \right\} - 1 \right| \tag{5.1}
\]

with \( G^1_1(x,t) := -\int_0^t (u^\varepsilon - u^0) \, d\tau, G^2_0(x,t) := \varepsilon \int_0^t (v^\varepsilon)^2 \, d\tau \) and \( G^3_0(x,t) := -\varepsilon \int_0^t v^\varepsilon_x \, d\tau \).

We next estimate \( G^1_1(x,t), G^2_0(x,t) \) and \( G^3_0(x,t) \). First, Theorem 2.1 gives

\[
\| G^1_1(x,t) \| \leq T\| u^\varepsilon - u^0 \|_{L^\infty([0,1] \times [0,T])} \leq C \varepsilon^{1/2}. \tag{5.2}
\]
Using Theorem 2.1, (4.15), Lemma 3.1-Lemma 3.3 and 0 < \varepsilon < 1, we estimate \( G_2^\varepsilon (x, t) \) as
\[
|G_2^\varepsilon (x, t)| \leq T \varepsilon \left( \|v^{I,0}\|_{L_\infty (0, T; H^1)}^2 + \|v^{B,0}\|_{L_\infty (0, T; H^1)}^2 + \|v^{b,0}\|_{L_\infty (0, T; H^1)}^2 + \varepsilon C \varepsilon^2 \right) \leq C \varepsilon. \tag{5.3}
\]
For any integer \( m \geq 2 \), similar arguments as deriving (4.13) entail that \( \|b_{2x}\|_{L^2 (0, T; L^\infty)} \leq C \varepsilon \), which along with the definition of \( \varepsilon \) in (4.2), (2.2), the Sobolev embedding inequality and Lemma 3.1-Lemma 3.6 and Lemma 4.4, leads to
\[
|G_3^\varepsilon (x, t)| \leq T^{1/2} \varepsilon \left( \|v^{I,0}\|_{L^2 (0, T; L^\infty)}^2 + \varepsilon^{-1/2} \|v^{B,0}\|_{L^2 (0, T; L^\infty)}^2 + \varepsilon^{-1/2} \|v^{b,0}\|_{L^2 (0, T; L^\infty)}^2 \right) + T^{1/2} \varepsilon\left[ \|v^{I,1}\|_{L^2 (0, T; L^\infty)}^2 + \|v^{B,1}\|_{L^2 (0, T; L^\infty)}^2 + \|v^{b,1}\|_{L^2 (0, T; L^\infty)}^2 \right]
\]
\[
+ T^{1/2} \varepsilon\left[ \|b_{2x}\|_{L^2 (0, T; L^\infty)}^2 + \varepsilon^{-1/2} \|v^{F}\|_{L^2 (0, T; H^1)}^2 \right]
\]
\[
\leq C \varepsilon^{3/4}. \tag{5.4}
\]
Collecting (5.2)-(5.4) and noticing that \( 0 < \varepsilon < 1 \), we end up with
\[
|G_1^\varepsilon (x, t) + G_2^\varepsilon (x, t) + G_3^\varepsilon (x, t)| \leq C \varepsilon^{1/2},
\]
for some positive constant \( C \) independent of \( \varepsilon \) (but dependent on \( T \)). Thus it follows from the Taylor expansion and 0 < \( \varepsilon < 1 \) that
\[
|e^{G_1^\varepsilon (x, t) + G_2^\varepsilon (x, t) + G_3^\varepsilon (x, t)} - 1| \leq \sum_{k=1}^{\infty} \frac{1}{k!} |G_1^\varepsilon (x, t) + G_2^\varepsilon (x, t) + G_3^\varepsilon (x, t)|^k \leq C \varepsilon^{1/2}. \tag{5.5}
\]
We proceed by employing (2.16) and find that
\[
0 < c^0 (x, t) = c_0 (x) e^{- \int_0^t \varepsilon v^0 (x, \tau) d\tau} \leq c_0 (x) \leq C_0, \tag{5.6}
\]
subject to the fact \( v^0 (x, t) \geq 0 \) for \( (x, t) \in [0, 1] \times [0, T] \). The combination of (5.1), (5.5) and (5.6) yields (2.17).

To prove (2.18), we use the transformation \( \varepsilon = - \frac{c^0}{c^\varepsilon} \), Theorem 2.1 and (2.17) and get
\[
\varepsilon^\varepsilon - c^0_c = - [v^\varepsilon c^\varepsilon - v^0 c^0_c]
\]
\[
= - [v^\varepsilon - v^0_c](c^\varepsilon - c^0_c)
\]
\[
= - [v^{I,0} + v^{B,0} + O(\varepsilon^{1/2})](c^\varepsilon + O(\varepsilon^{1/2})) + v^0 O(\varepsilon^{1/2})
\]
\[
= - \varepsilon^0 (v^{B,0} + v^{b,0}) + O(\varepsilon^{1/2}),
\]
which implies (2.18) and completes the proof of Theorem 2.2.

\[\Box\]

6. Appendix

In this section, we shall show the derivation of (2.3)-(2.12) by the method of matched asymptotic expansions. The same approach has been used in appendix of [26] to determine the thickness of boundary layers, where for the boundary layer profiles, only the equations on the leading-order left boundary layer profile \( (v^{B,0}, u^{B,1}) \) has been obtained. Here we carry out further procedures to deduce the equations (2.6)-(2.12) for \( (v^{b,0}, u^{b,1}) \) and the higher-order profiles. For brevity, we shall just outline the procedures that have not been demonstrated in [26].

Step 1. Initial-boundary conditions. Upon the substitution of (2.1) into the initial and boundary conditions in (1.5) and following the arguments in [26, Appendix, Step 2], one gets the initial conditions
\[
u^{I,0} (x, 0) = u_0 (x), \quad u^{B,0} (z, 0) = u^{b,0} (\xi, 0) = 0, \tag{6.1}
\]
and for \( j \geq 1 \)
\[
u^{I,j} (x, 0) = u^{B,j} (z, 0) = u^{b,j} (\xi, 0) = 0, \quad \nu^{I,j} (x, 0) = u^{B,j} (z, 0) = u^{b,j} (\xi, 0) = 0. \tag{6.2}
\]
The boundary conditions are given by
\begin{align}
\bar{u} &= u^{I,0}(0, t) + u^{B,0}(0, t), \quad \bar{v} = u^{I,0}(1, t) + u^{b,0}(0, t), \\
\tilde{v} &= v^{I,0}(0, t) + v^{B,0}(0, t), \quad \tilde{v} = v^{I,0}(1, t) + v^{b,0}(0, t),
\end{align}
and with \( j \geq 1 \) that
\begin{align}
u^{I,j}(0, t) + u^{B,j}(0, t) &= 0, \quad u^{I,j}(1, t) + u^{b,j}(0, t) = 0, \\
v^{I,j}(0, t) + v^{B,j}(0, t) &= 0, \quad v^{I,j}(1, t) + v^{b,j}(0, t) = 0.
\end{align}

**Step 2. Equations for \( u^{I,j} \), \( u^{B,j} \) and \( v^{b,j} \).** For profiles of \( u^{I,j} \), \( u^{B,j} \) we first employ the argument of [26, Appendix, Step 3] to derive
\begin{equation}
u^{I,j} - \sum_{k=0}^{j} (u^{I,k} v^{I,j-k})_x = u^{I,j}_{xx}, \quad \text{for } j \geq 0
\end{equation}
and
\begin{equation}
\sum_{j=2}^{\infty} \varepsilon^{j/2} \tilde{G}_j(z, t) = 0,
\end{equation}
with
\begin{align}
\tilde{G}_{-2} &= -u^{B,0}_{zz}, \\
\tilde{G}_{-1} &= -u^{I,0}(0, t)v^{B,0}_z - v^{I,0}(0, t)u^{B,0}_z - (u^{B,0} v^{B,0})_z - u^{B,1}_{zz}, \\
\tilde{G}_0 &= u^{B,0}_z - u^{B,0} v^{I,0}(0, t) - u^{I,0}(0, t) + u^{B,0} v^{B,1}_z - u^{I,1}(0, t) + u^{B,1} v^{B,0}_z - u^{I,0}(0, t) v^{B,0} \\
&\quad - u^{B,0}_z (v^{I,1}(0, t) + v^{B,1}) - u^{B,1}_z (v^{I,0}(0, t) + v^{B,0}) - u^{B,2} -zu^{I,0}(0, t) v^{B,0} - zv^{I,0}(0, t) u^{B,0}_z,
\end{align}
where \( \tilde{G}_j = 0 \) for \( j \geq -2 \). In particular \( \tilde{G}_{-2} = 0, \tilde{G}_{-1} = 0 \) and \( \tilde{G}_0 = 0 \) along with integration over \((z, \infty)\) entail that
\begin{align}
u^{B,0}(z, t) &= 0, \quad \text{for } (z, t) \in [0, \infty) \times [0, T], \\
u^{B,1}_z &= -u^{I,0}(0, t) v^{B,0} = -\bar{u} v^{B,0},
\end{align}
and
\begin{align}
u^{B,2}_{zz} &= -u^{I,0}(0, t) v^{B,1}_z - u^{I,1}(0, t) + u^{B,1} v^{B,0}_z \\
&\quad - u^{I,0}(0, t)v^{B,0}_z - u^{B,1}_z (v^{I,0}(0, t) + v^{B,0}) - zu^{I,0}(0, t)v^{B,0}.
\end{align}
Then integrating (6.9) with respect to \( z \) twice, one finds that
\begin{equation}
u^{B,2} = \bar{u} \int_{z}^{\infty} v^{B,1}(y, t) dy - \int_{z}^{\infty} \int_{y}^{\infty} \Phi(s, t) ds dy,
\end{equation}
with \( \Phi(z, t) := (u^{I,1}(0, t) + u^{B,1}) v^{B,0}_z + u^{I,0}(0, t)v^{B,0}_z + u^{B,1} (v^{I,0}(0, t) + v^{B,0}) + zu^{I,0}(0, t) v^{B,0}.
\)

For the right boundary-layer profiles \( v^{b,j} \), we modify the approach (detailed in [26, Appendix, Step 3]) in deriving (6.6) by neglecting the left boundary-layer profiles \( u^{B,j}, v^{B,j} \) in (2.1) and substituting the remaining terms into the first equation of (1.5), then subtracting (6.5) and applying the Taylor expansion (at \( x = 1 \)) to the remaining \( u^{I,j}, v^{I,j} \) in the resulting equation, to derive an expression similar to (6.6)
\begin{equation}
\sum_{j=2}^{\infty} \varepsilon^{j/2} \tilde{F}_j(x, t) = 0,
\end{equation}
where \( \tilde{F}_j \) is defined as \( \tilde{G}_j \) in (6.6) by replacing \((u^{B,k}, v^{B,k}) \) with \((u^{b,k}, v^{b,k}) \), \((u^{I,k}, v^{I,k})(0, t) \) with \((u^{I,k}, v^{I,k})(1, t) \) and \( z \) with \( \xi \), for \( k \in \mathbb{N} \). Hence, we deduce from \( \tilde{F}_{-2} = 0, \tilde{F}_{-1} = 0 \) and \( \tilde{F}_0 = 0 \) that
\begin{align}
v^{b,0}(\xi, t) &= 0, \quad \text{for } (\xi, t) \in (-\infty, 0] \times [0, T],
\end{align}
and
\begin{align}
u^{b,1}_\xi &= -\bar{u} v^{b,0},
\end{align}
\begin{align}
u^{b,0}(\xi, t) &= 0, \quad \text{for } (\xi, t) \in (-\infty, 0] \times [0, T],
\end{align}
and
\begin{align}
u^{b,1}_\xi &= -\bar{u} v^{b,0},
\end{align}
and
\[ u^{b,2}_{\xi\xi} = -u^{f,0}(1,t)v^{b,1}_\xi - (u^{f,1}(1,t) + u^{b,1})v^{b,0}_\xi \\
- u^{f,0}_x(1,t)v^{b,0} - u^{b,1}(u^{f,0}(1,t) + v^{b,0}) - \xi u^{f,0}_x(1,t)v^{b,0}. \] (6.13)

Thus
\[ u^{b,2} = \bar{u} \int_{\xi}^{-\infty} v^{b,1}(y,t) dy - \int_{\xi}^{-\infty} \int_{y}^{-\infty} \Psi(s,t) dsdy, \] (6.14)

with \( \Psi(\xi,t) := (u^{f,1}(1,t) + u^{b,1})v^{b,0}_\xi + u^{f,0}_x(1,t)v^{b,0} + u^{b,1}(u^{f,0}(1,t) + v^{b,0}) + \xi u^{f,0}_x(1,t)v^{b,0}. \)

**Step 3. Equations for** \( u^{I,j} \), \( v^{B,j} \) **and** \( v^{b,j} \). Applying the above arguments in Step 2 to the second equation of (1.5), we have for the outer layer profiles \( u^{I,j} \) that
\[ \begin{cases} 
u^{I,0}_t - u^{I,0}_x = 0, \\
\nu^{I,1}_t - u^{I,1}_x = 0, \\
\nu^{I,j}_t + 2 \sum_{k=0}^{j-2} \nu^{I,k}_x \nu^{I,j-2-k}_x - u^{I,j}_x - v^{I,j}_x = 0, \quad \text{for } j \geq 2 
\end{cases} \] (6.15)

and for the left boundary-layer profiles \( v^{B,j} \) that
\[ \begin{cases} -u^{B,0}_z = 0, \\
v^{B,1}_t - u^{B,1}_x - v^{B,0}_z = 0, \\
v^{B,1}_t + 2(u^{f,0}(0,t) + v^{B,0})v^{B,0}_z - u^{B,2}_z - v^{B,1}_z = 0, \\
\ldots \ldots \end{cases} \] (6.16)

Moreover, the right boundary-layer profiles \( v^{b,j} \) satisfy that
\[ \begin{cases} -u^{b,0}_{\xi} = 0, \\
v^{b,1}_t - u^{b,1}_{\xi} - v^{b,0}_{\xi} = 0, \\
v^{b,1}_t + 2(u^{f,0}(1,t) + v^{b,0})v^{b,0}_\xi - u^{b,2}_{\xi} - v^{b,1}_{\xi} = 0, \\
\ldots \ldots \end{cases} \] (6.17)

Finally, we collect the results obtained in Step 1 to Step 3 to derive the initial boundary value problems (2.3)-(2.12) given in section 2. First, from (6.5) with \( j = 0 \), (6.15), (6.1), (6.3), (6.7) and (6.11), we get (2.3). Combining (6.8), (6.16), (6.1) and (6.3), one gets (2.4)-(2.5). Similarly equations (6.12), (6.17), (6.1) and (6.3) lead to (2.6)-(2.7). Moreover, (6.5) with \( j = 1 \), (6.15), (6.2) and (6.4) give rise to (2.8), and equations (2.9)-(2.10) come from (6.10), (6.16), (6.2) and (6.4). Finally (2.11)-(2.12) follow from (6.14), (6.17), (6.2) and (6.4).

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