Additive rank-one preservers between spaces of rectangular matrices

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Abstract

Suppose $\mathbf{F}$ is a field and $m, n, p, q$ are positive integers. Let $M_{mn}(\mathbf{F})$ be the set of all $m \times n$ matrices over $\mathbf{F}$, and let $M^1_{mn}(\mathbf{F})$ be its subset consisting of all rank-one matrices. A map $\phi : M_{mn}(\mathbf{F}) \to M_{pq}(\mathbf{F})$ is said to be an additive rank-one preserver if $\phi(M^1_{mn}(\mathbf{F})) \subseteq M^1_{pq}(\mathbf{F})$ and $\phi(A + B) = \phi(A) + \phi(B)$ for any $A, B \in M_{mn}(\mathbf{F})$. This paper describes the structure of all additive rank-one preservers from $M_{mn}(\mathbf{F})$ to $M_{pq}(\mathbf{F})$.

Keywords Field; Matrix space; Rank one; Additive preserver.

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1 Introduction

In order to state precisely this article, we first introduce some concepts and fix the notation.

Suppose $\mathbf{F}$ is a field and $\mathbf{F}^* = \mathbf{F} \setminus \{0\}$. For positive integers $m$ and $n$, let $M_{mn}(\mathbf{F})$ be the set of all $m \times n$ matrices over $\mathbf{F}$, and let $M_n(\mathbf{F}) = M_{nn}(\mathbf{F})$ and $\mathbf{F}^n = M_{11}(\mathbf{F})$. Denote by $M_{mn}^r(\mathbf{F})$ the subset of $M_{mn}(\mathbf{F})$ consisting of all rank-$r$ matrices, and by $GL_n(\mathbf{F})$ the subset of $M_n(\mathbf{F})$ consisting of all nonsingular matrices. We use $\langle m \rangle$ to represent the set $\{1, 2, \ldots, m\}$. We denote by $E_{ij}$ the matrix with 1 in the $(i,j)$th entry and 0 elsewhere. For any matrix $B$, let $N(B)$, $B^T$ and $B^{-1}$ be the kernel space, transpose and inverse of $B$, respectively. For a map $\rho : \mathbf{F} \to \mathbf{F}$ and a matrix $A = [a_{ij}]$ over $\mathbf{F}$, we denote by $A^\rho$ the matrix $[\rho(a_{ij})]$. Symbol $\oplus$ denotes the usual direct sum of matrices.

An operator $\phi : M_{mn}(\mathbf{F}) \to M_{pq}(\mathbf{F})$ is said to be additive if $\phi(A + B) = \phi(A) + \phi(B)$ for any $A, B \in M_{mn}(\mathbf{F})$, linear if it is additive and satisfies $\phi(aA) = a\phi(A)$ for any $a \in \mathbf{F}$ and $A \in M_{mn}(\mathbf{F})$, and a rank-one preserver if $\phi(M_{mn}^1(\mathbf{F})) \subseteq M_{pq}^1(\mathbf{F})$.

In the recent several decades, characterizing linear/additive maps on spaces of matrices or operators that preserve certain properties has been an active area of research (see [5, 8, 11, 15] and the references therein). These are usually called linear/additive preserver problems in the literature. One of the most basic problems in linear/additive preserver problems is rank-one preserver problem, since some other questions about preservers have been solved with the help of rank-one preservers (see [1–4] and the references therein). Here we mention only partial results of rank-one preservers on spaces of rectangular matrices, which are closely related to this article. Marcus and Moyls [9] and Minc [10] described the structure of those linear rank-one preservers from $M_{mn}(\mathbf{F})$ into itself when $\mathbf{F}$ is any algebraically closed field of characteristic 0. Their difference lies in: Marcus and Moyls used multilinear algebra techniques, and Minc used only elementary matrix theory. Lim [7] characterized all invertible linear rank-one preservers
from $M_{mn}(F)$ into itself when $F$ is any field. Waterhouse [14] generalized the result of Lim to commutative rings with unit, but the invertibility assumption was still needed. Recently, Li et al. [6] obtained the following theorem (There is only a formal distinction to [6, Theorem 2.1]; they are in fact the same.) without the invertibility assumption.

**Theorem 1** Suppose $F$ is any field and $f : M_{mn}(F) \rightarrow M_{pq}(F)$ is a linear rank-one preserver. Then $f$ has one of the following four forms:

**(I)** $p \geq m \geq 2$, $q \geq n \geq 2$ and $f(A) = X(A \oplus 0)Y$ for any $A \in M_{mn}(F)$, where $X \in GL_p(F)$ and $Y \in GL_q(F)$.

**(II)** $p \geq n \geq 2$, $q \geq m \geq 2$ and $f(A) = X(A^T \oplus 0)Y$ for any $A \in M_{mn}(F)$, where $X$ and $Y$ are defined as in (I).

**(III)** $f(A) = \xi(A)\gamma^T$ for any $A \in M_{mn}(F)$, where $\gamma \in F^q \setminus \{0\}$ and $\xi : M_{mn}(F) \rightarrow F^p$ is a linear map such that $0 \notin \xi(M_{mn}^1(F))$.

**(IV)** $f(A) = \beta\eta(A)^T$ for any $A \in M_{mn}(F)$, where $\beta \in F^p \setminus \{0\}$ and $\eta : M_{mn}(F) \rightarrow F^q$ is a linear map such that $0 \notin \eta(M_{mn}^1(F))$.

Inspired by these works mentioned above, in this article we characterize the additive rank-one preservers from $M_{pq}(F)$ to $M_{np}(F)$ over any field $F$, i.e., investigating the following theorem.

**Theorem 2** For any field $F$, a map $\phi : M_{mn}(F) \rightarrow M_{pq}(F)$ is an additive rank-one preserver if and only if one of the following holds.

**(i)** There are $P \in M_{pm}(F)$, $Q \in M_{nq}(F)$ and an injective field endomorphism $\delta$ on $F$ such that $N(P) \cap \{x^\delta | x \in F^m\} = \{0\}$ and $N(Q^T) \cap \{y^\delta | y \in F^n\} = \{0\}$, and $\phi$ has the form $A \mapsto PA^\delta Q$. 

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(ii) There are $P \in M_{pn}(F)$, $Q \in M_{mq}(F)$ and an injective field endomorphism $\delta$ on $F$ such that $N(P) \cap \{x^\delta | x \in F^n\} = \{0\}$ and $N(Q^T) \cap \{y^\delta | y \in F^m\} = \{0\}$, and $\phi$ has the form $A \mapsto P(A^\delta)^T Q$.

(iii) There are nonzero $v \in F^q$ and an additive map $\mu : M_{mn}(F) \rightarrow F^p$ such that $0 \notin \mu(M_{mn}(F))$ and $\phi$ has the form $A \mapsto \mu(A)v^T$.

(iv) There are nonzero $u \in F^p$ and an additive map $\nu : M_{mn}(F) \rightarrow F^q$ such that $0 \notin \nu(M_{mn}(F))$ and $\phi$ has the form $A \mapsto u\nu(A)^T$.

The proof of Theorem 2 will be shown in the next section. Now we remark Theorem 2 as follows:

1. When $p \geq m+n-1$, [6, Proposition 2.3] provided an example of linear rank-one preserver of the form (iv). Here, we also give another example of linear rank-one preserver of the form (iv) when $F$ is a subfield of $R$ of all real numbers. Let $k$ be a positive integer, and let $f : M_{2,2k}(F) \rightarrow M_{1,2k}(F)$ be defined by

$$f : \sum_{i \in \langle 2 \rangle} \sum_{j \in \langle 2k \rangle} a_{ij} E_{ij} \mapsto \sum_{j \in \langle k \rangle} \left((a_{1,2j-1} - a_{2,2j})E_{1,2j-1} + (a_{1,2j} + a_{2,2j-1})E_{1,2j}\right).$$

Clearly, $f$ is a linear rank-one preserver of the form (iv). We can construct similar examples for the form described in (iii).

2. Note that, for any positive integers $g$ and $h$, $R$ is isomorphic to $M_{gh}(R)$ when they are viewed as additive groups. If $\xi_1 : M_{mn}(R) \rightarrow R$ and $\xi_2 : R \rightarrow R^g$ are additive group isomorphisms, then $\xi_2 \circ \xi_1$ is an additive group isomorphism from $M_{mn}(R)$ to $R^g$. Hence there is a non-linear additive map $\mu$ (respectively, $\nu$) satisfying (iii) (respectively, (iv)).

3. If $F$ is some field that is not isomorphic to a proper subfield of itself (for example, $F$ is a finite field), then any injective field endomorphism $\delta$ on $F$ is a field automorphism.
Hence, \( \{ x^\delta \mid x \in F^* \} = F^* \). Then for any \( A \in M_{rs}(F) \), \( N(A) \cap \{ x^\delta \mid x \in F^* \} = \{0\} \) if and only if \( A \) has full column rank. Therefore, if the condition “\( F \) is not isomorphic to a proper subfield of itself” is added, the matrices \( P \) and \( Q \) defined in (i)/(ii) can be extended to some \( p \times p \) invertible matrix \( X \) and \( q \times q \) invertible matrix \( Y \), respectively. Thus, the form described in (i) of Theorem 2 can be reduced to: \( p \geq m \geq 2, q \geq n \geq 2 \) and \( A \mapsto X(A^\delta \oplus 0)Y \) with \( X \in GL_m(F) \) and \( Y \in GL_n(F) \). For the form described in (ii) of Theorem 2, we have the similar conclusion.

4. One also observe that Theorem 1 can be obtained by Theorem 2 by using similar argument. Indeed if we further assume that \( \phi \) is linear, then the injective field endomorphism \( \delta \) defined in (i)/(ii) must be linear too. Thus, \( \delta \) must be the identity map, and hence the set \( \{ x^\delta \mid x \in F^* \} = F^* \).

5. If some appropriate restrictions on \( \phi \) and \( F \) (for example, \( F = \mathbb{R} \) and \( \phi \) preserves rank-one matrices in both directions) were added to Theorem 2, then Theorem 2 can be obtained directly from [12, 13]. However, the general case of Theorem 2 is not a direct corollary of [12, 13].

2 The Proof of Theorem 2

In this section, we investigate Theorem 2.

Proof of the sufficiency part of Theorem 2. It is trivial that \( \phi \) is an additive rank-one preserver if (iii) or (iv) holds. Suppose \( \phi \) has the form described in (i). Clearly, \( \phi \) is an additive map. It remains to show that \( \phi \) is a rank-one preserver.

Note that for any \( A \in M_{mn}^1(F) \),

\[
\text{rank}(\phi(A)) = \text{rank}(PA^\delta Q) \leq \text{rank}A^\delta = 1.
\]
Therefore, $\phi(M_{mn}^1(\mathbf{F})) \subseteq M_{pq}^1(\mathbf{F}) \cup \{0\}$. It suffices to show that $\phi(A) \neq 0$ for all $A \in M_{mn}^1(\mathbf{F})$.

Since both the sets $N(P) \cap \{x^\delta | x \in \mathbf{F}^m\}$ and $N(Q^T) \cap \{y^\delta | y \in \mathbf{F}^n\}$ contain the zero vectors only, $Px^\delta$ and $Q^Ty^\delta$ are nonzero for all nonzero $x \in \mathbf{F}^m$ and $y \in \mathbf{F}^n$. Then $\phi(xy^T) = P(xy^T)^\delta Q = Px^\delta(y^\delta)^TQ \neq 0$ for all $xy^T \in M_{mn}^1(\mathbf{F})$.

Similarly, we show that $\phi$ is an additive rank-one preserver if $\phi$ has the form described in (ii). □

For the necessary part, we only need to prove the following two theorems.

**Theorem 3** Suppose $\mathbf{F}$ is any field, and $\phi : M_{mn}^1(\mathbf{F}) \rightarrow M_{pq}^1(\mathbf{F})$ is an additive rank-one preserver such that $\text{rank}(\phi(H) + \phi(G)) > 1$ for some $H, G \in M_{mn}^1(\mathbf{F})$. Then $\phi$ has the form either (i) or (ii) of Theorem 2.

**Theorem 4** Suppose $\mathbf{F}$ is any field, and $\phi : M_{mn}^1(\mathbf{F}) \rightarrow M_{pq}^1(\mathbf{F})$ is an additive rank-one preserver such that $\text{rank}(\phi(H) + \phi(G)) \leq 1$ for any $H, G \in M_{mn}^1(\mathbf{F})$. Then $\phi$ has the form either (iii) or (iv) of Theorem 2.

We first prove Theorem 4 as follows.

**Proof of Theorem 4.** For any $A \in M_{mn}^1(\mathbf{F})$, it follows from the definition of $\phi$ that $\phi(A) \in M_{pq}^1(\mathbf{F})$, and hence for each $A \in M_{mn}^1(\mathbf{F})$

$$\phi(A) = u_A v_A^T \quad (1)$$

for some nonzero $u_A \in \mathbf{F}^p$ and $v_A \in \mathbf{F}^q$. Let

$$u = u_{E_{11}} \quad \text{and} \quad v = v_{E_{11}}. \quad (2)$$

**Case 1** Suppose the maximum number of linearly independent elements in $\phi(M_{mn}^1(\mathbf{F}))$ is 1. Then $\phi(A)$ and $\phi(E_{11})$ are linearly dependent for any $A \in M_{mn}^1(\mathbf{F})$. Thus, by (1) and (2), $\phi(A) = x_A u v^T$ for every $A \in M_{mn}^1(\mathbf{F})$, where $x_A \in \mathbf{F}^*$. Since every matrix in $M_{mn}(\mathbf{F})$ can be
written as a sum of finitely many matrices in $M_{mn}^1(F)$, we derive from the additivity of $\phi$ that $\phi$ is of the form (iii)/(iv).

**Case 2** Suppose the maximum number of linearly independent elements in $\phi(M_{mn}^1(F))$ is greater than or equal to 2. Then there exists $B \in M_{mn}^1(F)$ such that $\phi(E_{11})$ and $\phi(B)$ are linearly independent. This, together with (1) and (2), implies that either

(a) $u$ and $u_B$ are linearly independent; or

(b) $v$ and $v_B$ are linearly independent.

When (a) holds, it follows from (1), (2) and rank($\phi(E_{11}) + \phi(B)$) $\leq 1$ that $v$ and $v_B$ are linearly dependent, i.e., there is $c_B \in F^*$ such that $v_B = c_B v$. This, together with (1), (2), rank($\phi(E_{11}) + \phi(A)$) $\leq 1$ and rank($\phi(B) + \phi(A)$) $\leq 1$ for any $A \in M_{mn}^1(F)$, implies that for any $A \in M_{mn}^1(F)$, $v$ and $v_A$ are linearly dependent. Thus, $v_A = c_A v$ for some $c_A \in F^*$. Furthermore, $\phi(A) = c_A u_A v^T$. From the arbitrariness of $A \in M_{mn}^1(F)$, we can write (1) as $\phi(A) = \mu(A) v^T$ for any $A \in M_{mn}^1(F)$, where $\mu(A) = c_A u_A \in F^p \setminus \{0\}$. Since every matrix in $M_{mn}(F)$ can be written as a sum of finitely many matrices in $M_{mn}^1(F)$, we derive from the additivity of $\phi$ that $\phi$ has the form (iii). Similarly, when (b) holds, one can conclude that $\phi$ has the form (iv).

Now we devote our attention to the proof of Theorem 3. This requires the following three lemmas.

**Lemma 1** ([16, Lemma 1]) Let $F$ be any field and $A, B \in M_{mn}^1(F)$ satisfying $A + B \in M_{mn}^2(F)$. Then there are $P \in GL_m(F)$ and $Q \in GL_n(F)$ such that $A = P E_{11} Q$ and $B = P E_{22} Q$.

**Lemma 2** ([17, Lemma 2]) Let $C = [c_{gh}] \in M_{mn}^1(F)$. If $C + x E_{ij} \in M_{mn}^1(F)$ for some $x \in F^*, i \in \langle m \rangle, j \in \langle n \rangle$. Then $C = \sum_{g=1}^m c_{gj} E_{gj}$ or $\sum_{h=1}^n c_{ih} E_{ih}$. 


Lemma 3 Suppose \( F \) is any field and \( \psi : M_2(F) \to M_{pq}(F) \) is an additive rank-one preserver. If \( \psi(I_2) \in M_{pq}^2(F) \), then there are \( X \in \text{GL}_p(F) \), \( Y \in \text{GL}_q(F) \) and an injective field endomorphism \( \delta \) on \( F \) such that \( \psi \) has the form

\[
A \mapsto X(A^\delta \oplus 0)Y \quad \text{or} \quad A \mapsto X((A^\delta)^T \oplus 0)Y.
\]

Indeed, it is equivalent to say that there are linearly independent pairs \( x_1, x_2 \in F^p \) and \( y_1, y_2 \in F^q \) such that either

(i) \( \psi(\lambda E_{ij}) = \delta(\lambda)x_i y_j^T \) for all \( \lambda \in F \), \( i, j \in \langle 2 \rangle \), or

(ii) \( \psi(\lambda E_{ij}) = \delta(\lambda)x_j y_i^T \) for all \( \lambda \in F \), \( i, j \in \langle 2 \rangle \).

Proof. Since \( \psi \) is an additive rank one preserver, we have \( \psi(E_{11}), \psi(E_{22}) \in M_{pq}^1(F) \). This, together with \( \psi(E_{11}) + \psi(E_{22}) = \psi(I_2) \in M_{pq}^2(F) \) and Lemma 1, implies that

\[
\psi(E_{11}) = PE_{11}Q, \quad \psi(E_{22}) = PE_{22}Q \tag{3}
\]

for some \( P \in \text{GL}_p(F) \) and \( Q \in \text{GL}_q(F) \).

Define a map \( \psi_0 : M_2(F) \to M_{pq}(F) \) by

\[
\psi_0(A) = P^{-1}\psi(A)Q^{-1}, \forall A \in M_2(F). \tag{4}
\]

Then \( \psi_0 \) is an additive rank-one preserver, and further, we can derive from (3) that

\[
\psi_0(E_{11}) = E_{11}, \quad \psi_0(E_{22}) = E_{22}. \tag{5}
\]

For any \( x \in F^* \), since \( xE_{12}, E_{11} + xE_{12}, E_{22} + xE_{12} \in M_2^1(F) \), it follows from (5) that \( \psi_0(xE_{12}), E_{11} + \psi_0(xE_{12}), E_{22} + \psi_0(xE_{12}) \in M_{pq}^1(F) \). By Lemma 2, there is \( a_x \in F^* \) such that \( \psi_0(xE_{12}) = a_xE_{12} \) or \( a_xE_{21} \). Since \( \psi_0 \) is an additive rank-one preserver and \( xE_{12} - yE_{12} \in M_2^1(F) \) for any distinct \( x, y \in F^* \), it is seen that either

\[
\psi_0(\lambda E_{12}) = \pi(\lambda)E_{12}, \forall \lambda \in F \tag{6}
\]
\[ \psi_0(\lambda E_{12}) = \pi(\lambda)E_{21}, \ \forall \lambda \in F, \]  
\[ \psi_0(\lambda E_{21}) = \mu(\lambda)E_{12}, \ \forall \lambda \in F \]  
\[ \psi_0(\lambda E_{21}) = \mu(\lambda)E_{21}, \ \forall \lambda \in F, \] 

where \( \pi : F \to F \) is an injective additive map. Similarly, either

\begin{align*}
\psi_0(\lambda E_{21}) &= \mu(\lambda)E_{21}, \ \forall \lambda \in F, \\
\psi_0(\lambda E_{21}) &= \mu(\lambda)E_{12}, \ \forall \lambda \in F, \\
\psi_0(\lambda E_{12}) &= \delta(\lambda)E_{11}, \ \forall \lambda \in F, \\
\psi_0(\lambda E_{22}) &= \kappa(\lambda)E_{22}, \ \forall \lambda \in F, \\
\psi_0(\lambda E_{22}) &= \kappa(\lambda)E_{22}, \ \forall \lambda \in F, \\
\end{align*}

where \( \mu : F \to F \) is an injective additive map.

**Case 1** Suppose (6) and (8) hold simultaneously. Then it follows from (5) that \( \psi_0(E_{11} + E_{12} + E_{21} + E_{22}) = (\pi(1) + \mu(1))E_{12} + E_{11} + E_{22} \in M_{pq}^2(F) \), which contradicts that \( E_{11} + E_{12} + E_{21} + E_{22} \in M_2^1(F) \) and \( \psi_0 \) is an additive rank-one preserver.

**Case 2** Suppose (7) and (9) hold simultaneously. By an argument similar to Case 1, one can derive a contradiction.

**Case 3** Suppose (6) and (9) hold simultaneously. For any \( \lambda \in F \), it follows from \( \text{rank}(\lambda E_{11} + E_{11}) \leq 1 \) and \( E_{12} + \lambda E_{11}, \ E_{21} + \lambda E_{11} \in M_2^1(F) \) that \( \text{rank}(\psi_0(\lambda E_{11}) + \psi_0(E_{11})) \leq 1 \) and \( \psi_0(E_{12}) + \psi_0(\lambda E_{11}), \ \psi_0(E_{21}) + \psi_0(\lambda E_{11}) \in M_{pq}^1(F) \). Using Lemma 2, (5), (6) and (9), we have

\[ \psi_0(\lambda E_{11}) = \delta(\lambda)E_{11}, \ \forall \lambda \in F, \]  
\[ \psi_0(\lambda E_{21}) = \mu(\lambda)E_{12}, \ \forall \lambda \in F, \]  
\[ \psi_0(\lambda E_{22}) = \kappa(\lambda)E_{22}, \ \forall \lambda \in F, \]  

where \( \delta : F \to F \) is an injective additive map with \( \delta(1) = 1 \). Similarly,

\[ \psi_0(\lambda E_{22}) = \kappa(\lambda)E_{22}, \ \forall \lambda \in F, \]  

where \( \kappa : F \to F \) is an injective additive map with \( \kappa(1) = 1 \). Because of \( \lambda E_{11} + \lambda E_{12} + E_{21} + E_{22}, \ E_{11} + \lambda E_{12} + E_{21} + \lambda E_{22} \in M_2^1(F) \), it follows from (6) and (9)—(11) that \( \delta(\lambda)E_{11} + \pi(\lambda)E_{12} + \mu(1)E_{21} + E_{22}, \ E_{11} + \pi(\lambda)E_{12} + \mu(1)E_{21} + \kappa(\lambda)E_{22} \in M_{pq}^1(F) \), and hence

\[ \delta(\lambda) = \kappa(\lambda) = \pi(\lambda)\mu(1), \ \forall \lambda \in F. \]
Similarly,
\[ \delta(\lambda) = \kappa(\lambda) = \pi(1)\mu(\lambda), \forall \lambda \in F. \quad (13) \]
If we denote \( U = \mu(1)^{-1} \oplus I_{p-1} \) and \( V = \mu(1) \oplus I_{q-1} \), then it is easy to verify from \( \delta(1) = 1 \), (6) and (9)—(13) that
\[ \psi_0(\lambda E_{ij}) = \delta(\lambda)U E_{ij}V, \forall \lambda \in F, \ i, j \in \langle 2 \rangle. \quad (14) \]

For any \( a, b \in F \), because of \( abE_{11} + aE_{12} + bE_{21} + E_{22} \in M_{2}^{1}(F) \), it follows from (14) that \( \delta(ab)E_{11} + \delta(a)E_{12} + \delta(b)E_{21} + E_{22} \in M_{pq}^{1}(F) \), and hence \( \delta(ab) = \delta(a)\delta(b) \). Since \( \delta : F \to F \) is an injective additive map with \( \delta(1) = 1 \), it can be concluded that \( \delta \) is an injective field endomorphism on \( F \). This, together with (14) and the additivity of \( \psi_0 \), \( \psi_0 \) has the form \( A \mapsto U(A^\delta \oplus 0)V \). Thus, with (4), \( \psi \) has the same form too.

**Case 4** Suppose (7) and (8) hold simultaneously. Then by an argument similar to Case 3, \( \psi \) has the form \( A \mapsto U((A^\delta)^T \oplus 0)V \). \( \blacksquare \)

Based on the above preparations, one can prove Theorem 3 as follows.

**Proof of Theorem 3.** Suppose there are \( H, G \in M_{mn}^{1}(F) \) such that \( \text{rank}(\phi(H) + \phi(G)) > 1 \). Then \( \text{rank}(H + G) > 1 \). Since \( \text{rank}(H + G) \leq \text{rank}H + \text{rank}G = 2 \), we conclude that \( H + G \in M_{mn}^{2}(F) \). By singular decomposition, there are \( U \in GL_m(F) \) and \( V \in GL_n(F) \) such that \( H + G = U(I_2 \oplus 0)V \).

Define \( \psi_1 : M_2(F) \to M_{pq}(F) \) by \( \psi_1(B) = \phi(U(B \oplus 0)V) \) for all \( B \in M_2(F) \). Then \( \psi_1 \) is an additive rank-one preserver. Furthermore,
\[ \psi_1(I_2) = \phi(U(I_2 \oplus 0)V) = \phi(H + G) = \phi(H) + \phi(G), \]
and hence \( \text{rank}\psi_1(I_2) \leq \text{rank}\phi(H) + \text{rank}\phi(G) \). This, together with \( \phi(M_{mn}^{1}(F)) \subseteq M_{pq}^{1}(F) \) and \( \text{rank}\psi_1(I_2) = \text{rank}(\phi(H) + \phi(G)) > 1 \), implies that \( \psi_1(I_2) \) has rank two. Then by Lemma 3,
there are \( P \in GL_p(F) \), \( Q \in GL_q(F) \) and an injective field endomorphism \( \delta \) on \( F \) such that

\[
\phi(U(B \oplus 0)V) = \psi_1(B) = P(B^\delta \oplus 0)Q \quad \text{for all } B \in M_2(F)
\]
or

\[
\phi(U(B \oplus 0)V) = \psi_1(B) = P((B^\delta)^T \oplus 0)Q \quad \text{for all } B \in M_2(F).
\]

Replacing \( \phi \) by the maps \( A \mapsto P^{-1}\phi(UAV)Q^{-1} \) or \( A \mapsto (P^{-1}\phi(UAV)Q^{-1})^T \), we may assume that

\[
\phi(B \oplus 0) = B^\delta \oplus 0 \quad \text{for all } B \in M_2(F).
\]

(15)

Now for any \( j \in \langle n \rangle \setminus \{1, 2\} \), since

\[
E_{21} + \phi(E_{2j}) = \phi(E_{21} + E_{2j}) \quad \text{and} \quad E_{22} + \phi(E_{2j}) = \phi(E_{22} + E_{2j})
\]

are rank one, we check that at least one of

\[
E_{11} + \phi(E_{2j}) = \phi(E_{11} + E_{2j}) \quad \text{and} \quad E_{12} + \phi(E_{2j}) = \phi(E_{12} + E_{2j})
\]

has rank two. Otherwise, we have \( \phi(E_{2j}) = 0 \), but this contradicts to \( \phi(M_{mn}^1(F)) \subseteq M_{pq}^1(F) \).

Let \( E_{1k} \), where \( k = 1 \) or \( 2 \), be the matrix for which \( \phi(E_{1k} + E_{2j}) \) has rank two. We define \( \psi_2 : M_2(F) \to M_{pq}(F) \) by

\[
\psi_2 \left( \begin{array}{cc} b_{11} & b_{12} \\ b_{21} & b_{22} \end{array} \right) = \phi \left( b_{11}E_{1k} + b_{12}E_{1j} + b_{21}E_{2k} + b_{22}E_{2j} \right) \quad \text{for all } \left( \begin{array}{cc} b_{11} & b_{12} \\ b_{21} & b_{22} \end{array} \right) \in M_2(F).
\]

Then \( \psi_2 \) is an additive rank-one preserver and \( \psi_2(I_2) = \phi(E_{1k} + E_{2j}) \) has rank two. From (15), we have

\[
\psi_2(\lambda E_{11}) = \phi(\lambda E_{1k}) = \delta(\lambda)e_1e_k^T \quad \text{and} \quad \psi_2(\lambda E_{21}) = \phi(\lambda E_{2k}) = \delta(\lambda)e_2e_k^T \quad \text{for all } \lambda \in F.
\]

Then by Lemma 3, there is \( y_j \in F^q \) such that \( y_j \) and \( e_k \) are linearly independent and

\[
\phi(\lambda E_{1j}) = \psi_2(\lambda E_{12}) = \delta(\lambda)e_1y_j^T \quad \text{and} \quad \phi(\lambda E_{2j}) = \psi_2(\lambda E_{22}) = \delta(\lambda)e_2y_j^T \quad \text{for all } \lambda \in F.
\]
Let $Y^T = [y_1 \cdots y_n]$ with $y_1 = e_1$ and $y_2 = e_2$. Then with (15), we have
\[
\phi (\lambda E_{ij}) = \delta(\lambda)e_i e_j^T Y \quad \text{for all } \lambda \in \mathbb{F}, \ i \in \langle 2 \rangle \text{ and } j \in \langle n \rangle.
\] (16)

Now for any $i \in \langle m \rangle \setminus \{1, 2\}$ and $j \in \langle n \rangle \setminus \{k\}$, we check that at least one of $\phi(E_{1k} + E_{ij})$ and $\phi(E_{2k} + E_{ij})$ has rank two. Let $E_{lk}$, where $l = 1$ or $2$, be the matrix for which $\phi(E_{lk} + E_{ij})$ has rank two. Similarly, with Lemma 3, (16) and by considering the map $\psi_3 : M_2(\mathbb{F}) \to M_{pq}(\mathbb{F})$ defined by
\[
\psi_3 \left( \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \right) = \phi \left( b_{11}E_{ik} + b_{12}E_{ij} + b_{21}E_{ik} + b_{22}E_{ij} \right) \quad \text{for all} \quad \left( \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \right) \in M_2(\mathbb{F}),
\]
we conclude that there is $x_i \in \mathbb{F}^p$ such that $x_i$ and $e_i$ are linearly independent and
\[
\phi (\lambda E_{ik}) = \psi_3 (\lambda E_{21}) = \delta(\lambda)x_i x_k^T Y \quad \text{and} \quad \phi (\lambda E_{ij}) = \psi_3 (\lambda E_{22}) = \delta(\lambda)x_i e_j^T Y \quad \text{for all} \quad \lambda \in \mathbb{F}.
\]
Let $X = [x_1 \cdots x_m]$ with $x_1 = e_1$ and $x_2 = e_2$. Then with (16), we have
\[
\phi (\lambda E_{ij}) = \delta(\lambda)Xe_i e_j^T Y = \delta(\lambda)XE_{ij}Y \quad \text{for all} \quad \lambda \in \mathbb{F}, \ i \in \langle m \rangle \text{ and } j \in \langle n \rangle.
\]
As $\phi$ is additive, we deduce that $\phi (A) = XA^\delta Y$ for all $A \in M_{mn}(\mathbb{F})$.

Finally, for any nonzero $x \in \mathbb{F}^m$ and $y \in \mathbb{F}^n$, $\phi (xy^T) = Xx^\delta(y^\delta)^T Y \neq 0$ as $\phi$ is a rank-one preserver. Therefore, $Xx^\delta$ and $Y^Ty^\delta$ are nonzero for all nonzero $x \in \mathbb{F}^m$ and $y \in \mathbb{F}^n$, i.e.,
\[
N(X) \cap \{x^\delta \mid x \in \mathbb{F}^m\} = \{0\} \quad \text{and} \quad N(Y^T) \cap \{y^\delta \mid y \in \mathbb{F}^n\} = \{0\}.
\]

### 3 Concluding remarks

This article characterized the additive rank-one preservers from $M_{mn}(\mathbb{F})$ to $M_{pq}(\mathbb{F})$ over any field $\mathbb{F}$ without the surjectivity assumption. As shown in [1–4], some preserver problems on matrix spaces can be reduced to rank-one preserver problems. This provides the possibility for removing the surjectivity assumption of some results on additive preserver problems.
Further work is to solve some preserver problems between spaces of rectangular matrices by reducing them to the results obtained in this article.

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