Every invertible matrix is diagonally equivalent to a matrix with distinct eigenvalues

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Abstract

We show that for every invertible $n \times n$ complex matrix $A$ there is an $n \times n$ diagonal invertible $D$ such that $AD$ has distinct eigenvalues. Using this result, we affirm a conjecture of Feng, Li, and Huang that an $n \times n$ matrix is not diagonally equivalent to a matrix with distinct eigenvalues if and only if it is singular and all its principal minors of size $n - 1$ are zero.

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1 Introduction

Denote by $M_n$ the set of $n \times n$ complex matrices. In [1], the authors pointed out that matrices with distinct eigenvalues have many nice properties. They then raised the question whether every invertible matrix in $M_n$ is diagonally equivalent to a matrix with distinct eigenvalues, and conjectured that a matrix in $M_n$ is not diagonally equivalent to a matrix with distinct eigenvalues if and only if it is singular and every principal minor of size $n - 1$ is zero. They provided a proof for matrices in $M_n$ with $n \leq 3$, and demonstrated the complexity of the problem for matrices in $M_4$ using their approach. In this note, we affirm their conjecture by proving the following theorem.

Theorem 1.1 Suppose $A \in M_n$ is invertible. There is an invertible diagonal $D \in M_n$ such that $AD$ has distinct eigenvalues.

Once this result is proved, we have the following corollary.
Corollary 1.2 Let $A \in M_n$. The following are equivalent.

(a) $A$ is not diagonally equivalent to a matrix with distinct eigenvalues.

(b) There is no diagonal matrix $D$ such that $AD$ has distinct eigenvalues.

(c) The matrix $A$ is singular and all principal minors of size $n-1$ are zero.

Proof. The implication (a) $\Rightarrow$ (b) is clear. Suppose condition (c) does not hold. Then either $A$ is invertible or $A$ has an invertible principal submatrix of size $n-1$. Assume the former case holds. There is an invertible diagonal matrix $D$ such that $AD$ has distinct eigenvalues by Theorem 1.1. If the latter case holds, we may assume without loss of generality that the leading principal submatrix $A_1 \in M_{n-1}$ is invertible. By Theorem 1.1, there is an invertible diagonal matrix $D_1 \in M_{n-1}$ such that $A_1D_1$ has distinct (nonzero) eigenvalues. Let $D = D_1 \oplus [0]$. Then $AD$ has distinct eigenvalues including zero as an eigenvalue. Thus, (b) cannot hold. So, we have proved (a) $\Rightarrow$ (b) $\Rightarrow$ (c).

Recall that the characteristic polynomial of a matrix $B \in M_n$ has the form $\det(xI_n - B) = x^n + b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + \cdots + b_1x + b_0$, where $(-1)^jb_{n-j}$ is the sum of $j \times j$ principal minors of $B$. Suppose condition (c) holds. Since the principal minors of $D_1AD_2$ are scalar multiples of the corresponding principal minors of $A$, then $D_1AD_2$ has characteristic polynomial of the form $\det(xI_n - D_1AD_2) = x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2$ so that $0$ is a root with multiplicity at least two. Thus, $D_1AD_2$ cannot have $n$ distinct eigenvalues. So, the implication (c) $\Rightarrow$ (a) is proved.

Note that the set of diagonal matrices is an $n$-dimensional subspace in $M_n$. We can extend Theorem 1.1 to the following.

Corollary 1.3 Suppose $V$ is a subspace of matrices in $M_n$.

(a) If there are invertible matrices $R$ and $S$ such that $RVS = \{RXS : X \in V\}$ contains the subspace of diagonal matrices, then for any invertible $A \in M_n$ there is $X \in V$ such that $AX$ has distinct eigenvalues.

(b) If there are invertible matrices $R$ and $S$ such that $RVS$ has zero first row and zero last column for every $X \in V$, then $A = SR$ is invertible and $AX$ is similar to $RXS$ which cannot have distinct eigenvalues for any $X \in V$.

Proof. (a) Suppose $A$ is invertible. Then there is a diagonal matrix $D$ such that $S^{-1}AR^{-1}D$ has distinct eigenvalues by Theorem 1.1. Set $X = R^{-1}DS^{-1} \in V$. Notice that $AX$ has distinct eigenvalues as $S^{-1}(AX)S = S^{-1}(AR^{-1}DS^{-1})S = (S^{-1}AR^{-1})D$.

Assertion (b) can be verified readily.

2 Proof of Theorem 1.1

We will prove Theorem 1.1 by induction on $n$. The result is clear if $A \in M_1$. Assume that the result holds for all $k \times k$ invertible matrices with $1 \leq k < n$. Suppose $A \in M_n$ is invertible. We consider two cases.
Case 1. If all \( k \times k \) principal minors of \( A \) are singular for \( k = 1, \ldots, n-1 \), then the characteristic polynomial of \( A \) has the form \( x^n - a_0 \) and has \( n \) distinct roots. So, the result holds with \( D = I_n \).

Case 2. Suppose \( A \) has an invertible \( k \times k \) principal minor. Without loss of generality, we may assume that \( A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \) such that \( A_{11} \in M_k \) is invertible for some \( 1 \leq k < n \). Then the Schur complement of \( A_{22} \) equals \( B = A_{22} - A_{21}A_{11}^{-1}A_{12} \) which is invertible; see [2, pp. 21-22].

By induction assumption, there are diagonal invertible \( D_1 \) \( D_2 \) \( D \) such that each of \( A_{11}D_1 \) and \( BD_2 \) has distinct nonzero eigenvalues, say, \( \lambda_1, \ldots, \lambda_k \) and \( \lambda_{k+1}, \ldots, \lambda_n \), respectively. Thus, \( A_{11}D_1 \) and \( BD_2 \) are diagonalizable and there are invertible \( S_1 \in M_k \) and \( S_2 \in M_{n-k} \) such that \( S_1A_{11}D_1S_1^{-1} = \Lambda_1 = \text{diag} (\lambda_1, \ldots, \lambda_k) \) and \( S_2BD_2S_2^{-1} = \Lambda_2 = \text{diag} (\lambda_{k+1}, \ldots, \lambda_n) \). Let \( D_{r,s} = rD_1 \oplus sD_2 \).

The proof is complete if one can find some subitable \( r \) and \( s \) so that \( AD_{r,s} \) has distinct eigenvalues. Notice that \( AD_{r,s} \) has the same eigenvalues as

\[
\tilde{A} = \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix} \begin{bmatrix} I_k & 0 \\ -A_{21}A_{11}^{-1} & I_{n-k} \end{bmatrix} \begin{bmatrix} I_k & 0 \\ A_{21}A_{11}^{-1} & I_{n-k} \end{bmatrix} \begin{bmatrix} S_1^{-1} & 0 \\ 0 & s^{-1}S_2^{-1} \end{bmatrix}
= \begin{bmatrix} r\Lambda_1 + sS_1A_{12}D_2A_{21}A_{11}^{-1}S_1^{-1} & S_1A_{12}D_2S_2^{-1} \\ s^2S_2BD_2A_{21}A_{11}^{-1}S_1^{-1} & s\Lambda_2 \end{bmatrix}.
\]

Denote by \( D(a,d) \) the closed disk in \( \mathbb{C} \) centered at \( a \) with radius \( d \geq 0 \). Suppose the \( k \times k \) matrix \( S_1A_{12}D_2A_{21}A_{11}^{-1}S_1^{-1} \) has diagonal entries \( \mu_1, \ldots, \mu_k \) and let

\[
d_1 = k\|S_1A_{12}D_2A_{21}A_{11}^{-1}S_1^{-1}\|, \quad d_2 = (n-k)\|S_1A_{12}D_2S_2^{-1}\|, \quad \text{and} \quad d_3 = k\|S_2BD_2A_{21}A_{11}^{-1}S_1^{-1}\|,
\]

where \( \| \cdot \| \) is the operator norm. By Geršgorin disk result (see [2, pp.344-347]), the eigenvalues of \( \tilde{A} \) must lie in the union of the \( n \) Geršgorin disks, which is a subset of the union of \( n \) disks

\[
D(r\lambda_1 + s\mu_1, sd_1 + d_2), \ldots, D(r\lambda_k + s\mu_k, sd_1 + d_2), D(s\lambda_{k+1}, s^2d_3), \ldots, D(s\lambda_n, s^2d_3).
\]

We can choose sufficiently large \( r > 0 \) and sufficiently small \( s > 0 \) so that these disks are disjoint, and hence \( \tilde{A} \) has \( n \) disjoint Geršgorin disks. Then \( \tilde{A} \) has distinct eigenvalues.

We thank Editor Zhan for sending us the two related references [3, 4]. In these papers, the author proved following. Suppose \( A \) is an \( n \times n \) matrix and \( a_1, \ldots, a_n \) are complex numbers. Then there is a diagonal matrix \( E \) such that \( A + E \) has eigenvalues \( a_1, \ldots, a_n \). Moreover, if all principal minors of \( A \) are nonzero, then there is a diagonal matrix \( D \) such that \( AD \) has eigenvalues \( a_1, \ldots, a_n \).

Note that the assumption on the principal minors of \( A \) is important in the second assertion. Obviously, if \( \det(A) = 0 \), then one cannot find diagonal \( D \) such that \( AD \) has \( n \) nonzero eigenvalues. Even if we remove this obvious obstacle and assume that \( A \) is invertible, one may not be able to find diagonal \( D \) so that \( AD \) has prescribed eigenvalues. For example, if \( \{E_{1,1}, E_{1,2}, \ldots, E_{n,n}\} \) is the standard basis for \( M_n \) and \( A = E_{1,2} + \cdots + E_{n-1,n} \), then the eigenvalues of \( AD \) always have the form \( z, zw, \ldots, zw^{n-1} \) for some \( z \in \mathbb{C} \), where \( w \) is the primitive \( n \)th root of unity.
It is interesting to determine the condition on $A$ so that for any complex numbers $a_1, \ldots, a_n$, one can find a diagonal $D$ such that $AD$ has $a_1, \ldots, a_n$ as eigenvalues.

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References


