PERFORMANCE COMPARISON OF DIGITAL IMPLEMENTATION OF ANALOG SYSTEMS

Guofeng Zhang, Xiang Chen, and Tongwen Chen

Abstract—A new controller discretization approach, the generalized bilinear transformation (GBT), is proposed in [1]. Given an analog controller \( K \), GBT generates a class of digital controllers \( K_{\text{SD}} \) parameterized by \( \alpha \in (-\infty, \infty) \). A geometric interpretation of GBT is first presented. Secondly, when the original analog feedback system is stable, a method is proposed to find the value of the parameter \( \alpha \) which provides upper bound of the sampling period guaranteeing closed-loop stability of the resulting sampled-data system. Thirdly, it is shown that step-tracking is preserved if the closed-loop sampled-data system is stable. Finally, two examples are used to demonstrate the strength of our digitization approach.

Keywords—controller discretization, generalized bilinear transformation (GBT), closed-loop stability, optimization.

I. INTRODUCTION

Computer control has entered every facet of our daily life; they usually appear in the form of embedded microprocessors in real-time control systems. Compared with analog control, computer control has many advantages. For example, it has no drifting phenomenon present in analog systems; it can achieve deadbeat control that is impossible in analog control; and more importantly, due to the use of computers, fairly complicated control algorithms can be implemented easily to meet high-demanding control performance specifications. A typical computer control system is a loop composed sequentially of a physical plant (usually in continuous time), an analog-to-digital (A/D) converter that normally contains samplers and quantizers, a digital controller and a digital-to-analog (D/A) converter containing certain hold circuits. So the digital controller is a fundamental ingredient of any computer control system.

The methods of designing digital controllers can be roughly classified into three types. The first type is the so-called discrete design, where a digital controller is designed for the discrete-time counterpart of the original analog plant. Therefore, inter-sample ripple is always a problem in this design method [2]. The intent to cope with inter-sample behavior motivates the other extreme: the sampled-data (SD) design, namely, design based on continuous lifting. Although this design method can deal with inter-sample behavior perfectly, its inherent design complexity hinders its practical applicability [3]. The emulation method can be regarded as a method in-between the two methods aforementioned. The emulation method consists of two steps. The first step is to design a (fictitious) continuous-time controller, we call this step the design step. The second step is to discretize the continuous-time controller to obtain a discrete-time controller. We call this step the implementation step. There are lots of methods supporting controller design in the continuous-time domain. However, there are merely several controller discretization approaches in the implementation step, Tustin, ZOH equivalent, the Euler method and the matched pole-zero method being the most popular ones [4]. Therefore, the emulation method usually goes like this: First design in the continuous-time domain, then try several controller discretization approaches such as those listed above. If the closed-loop performance is satisfactory, the job is done; otherwise, go back to redesign in the continuous-time domain. In other words, this situation makes the emulation method essentially a process of trial and error. With the aim of improving this situation, a new controller discretization — the Generalized Bilinear Transformation (GBT) is proposed in [1].

GBT provides a class of controller approximations parameterized by a parameter \( \alpha \in (-\infty, \infty) \). With the aid of GBT, the emulation method now works in the following way: First an analog controller is designed in the continuous-time domain following some physical sense, then an optimization problem is solved over \( \alpha \in (-\infty, \infty) \) to find a desired digital controller. In this manner, the emulation method becomes a systematic method and the search for an optimal digital controller is made possible. One may doubt if the resulting digital controller obtained via GBT works satisfactorily, this paper will show that design via GBT is effective in various contexts, ranging from stability, step tracking, to \( H_\infty \) control, etc.. It may even provide the same control performance as the SD design.

The problem of optimization-based controller discretization has been extensively investigated in the literature (e.g., [5], [6], [7], [8]). In those papers, given an analog controller, an optimization problem is solved to produce a digital controller which is optimal in a certain sense. Clearly, only one controller is obtained. Since every control system operates undoubtedly under a variety of uncertainties, it is often desirable to adjust control law to cope with real situations if necessary [9], i.e., it is better to have an adaptive controller. The parameter \( \alpha \) in GBT adds an extra degree of freedom to the control system concerned, therefore making the closed-
The remainder of this paper is organized as follows.

Section 2 introduces the generalized bilinear transformation (GBT) from a geometrical point of view.

Section 3 investigates the stability of sampled-data systems designed based on GBT. Assume that the original analog feedback system is stable. A method is proposed to find the suitable parameter $\alpha$ which gives upper bound of the sampling period guaranteeing the closed-loop stability of the resulting sampled-data system.

Section 4 focuses on the problem of step tracking. Suppose that the original analog system is stable and step-tracking, then the sampled-data system is step-tracking for all $\alpha \in (-\infty, \infty)$ if it is stable. An example shows that transient behavior can be improved considerably when $\alpha$ is suitably chosen.

Section 5 discusses an application of GBT. For a system studied in [10] which is reported that common digitization methods yield either non-stabilizing controllers or systems with very poor closed-loop performance, we will compare GBT and the controller re-design method proposed in [5] to demonstrate the strength of GBT.

Section 6 consists of some concluding remarks.

Finally, some words about notation. The norm symbol $\| \cdot \|$ represents the Euclidean norm if $\cdot$ is a vector or its largest singular value if $\cdot$ is a matrix; $\| \cdot \|_{\ell_p}$ is the $\ell_p$ norm if applied to a vector and $\ell_p$ induced norm if applied to a system. Following the convention, for a continuous-time controller $K$, $K(s)$ is used to denote its transfer function. A state-space realization of $K$ is denoted $(A_K, B_K, C_K, D_K)$ and we also define

$$
\begin{bmatrix}
A_K & B_K \\
C_K & D_K
\end{bmatrix} := D_K + C_K(sI - A_K)^{-1}B_K.
$$

The discrete-time case can be treated in a similar way.

II. THE GENERALIZED BILINEAR TRANSFORMATION

The (traditional) bilinear transformation is motivated by considering the trapezoidal approximation of an integrator. Given an integrator $1/s$ with input $u$ and output $y$, the trapezoidal approximation of $y(kh + h) = y(kh) + \int_{kh}^{kh+h} u(\tau)d\tau$ (1) using some other combination of $u(kh + h)$ and $u(kh)$. More specifically, consider

$$
y(kh + h) = y(kh) + h\left[\alpha u(kh + h) + (1 - \alpha)u(kh)\right],
$$

where $\alpha \in (-\infty, \infty)$. The transfer function of Eq. (2) is (in $z$ transform)

$$
y(z) = u(z) = \frac{\alpha z + (1 - \alpha)}{z - 1}.
$$

This motivates us to introduce the following generalized bilinear transformation (GBT)

$$
\frac{s}{1} = \frac{\alpha z + (1 - \alpha)}{z - 1},
$$

that is

$$
s = \frac{1}{h} \frac{z - 1}{\alpha z + (1 - \alpha)}.
$$

Therefore, under the generalized bilinear transformation, a finite-dimensional linear time-invariant (FDLTI) continuous-time transfer function $K(s)$ is mapped to $K_{gbt}(z)$, where

$$
K_{gbt}(z) := K\left[\frac{1}{h} \frac{z - 1}{\alpha z + (1 - \alpha)}\right]
$$

with $\alpha \in (-\infty, \infty)$. In terms of state-space data, bring in a minimal realization of $K(s)$, namely, $(A_K, B_K, C_K, D_K)$, it is straightforward to derive that $K_{gbt}(z)$ has a state-space model $(A_{Kgbt}, B_{Kgbt}, C_{Kgbt}, D_{Kgbt})$, where

$$
\begin{align*}
A_{Kgbt} &= \left(I - \alpha hA\right)^{-1} [I + (1 - \alpha)hA], \\
B_{Kgbt} &= \left(I - \alpha hA\right)^{-1} hB, \\
C_{Kgbt} &= C_K\left(I - \alpha hA\right)^{-1}, \\
D_{Kgbt} &= D_K + \alpha C_KB_{Kgbt}.
\end{align*}
$$

Remark 1: When $\alpha = 0$, 1/2, and 1, $K_{gbt}(z)$ is the forward Euler, Tustin and backward Euler approximation of $K(s)$ respectively. These approximations are illustrated geometrically in Fig. 1. In each case, the integral equals the area enclosed by the curve, the two vertical dotted lines and the horizontal axis. In (a), (b) and (c), the approximations equal the area enclosed by the three dotted lines and the horizontal axis. However, in (d), the approximation equals the difference between the positive area (above the horizontal axis) and the negative area (under the horizontal axis). One never met such an approximation in any calculus book. However, this approximation turns out to be quite useful in digital control.

III. STABILITY

The purpose of this section is to study the stability of sampled-data systems implemented via GBT.

Consider the continuous-time feedback control system $\Sigma_1$ as shown in Fig. 2. Suppose that both the plant $G$ and the controller $K$ are FDLTI. Furthermore, assume that $G$ is strictly proper and $K$ is proper. Bring in minimal state-space realizations:

$$
G(s) = \begin{bmatrix} A_G & B_G \\ C_G & 0 \end{bmatrix}, \quad K(s) = \begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix}.
$$


Fig. 1. (a): forward Euler, (b): backward Euler, (c): bilinear, (d): GBT

Let the state variables of $\alpha$ be $x_G$ and $x_K$ respectively. Moreover, define their compound $x_{G,K} := [x_G^T, x_K^T]^T$, where the superscript “$T$” is the transpose operation. Then $x_{G,K}$ is called exponentially stable if there exist positive constants $\mu$ and $\nu$ such that, for every initial time $t_0$ and initial state $x_{G,K}(t_0)$,

$$\|x_{G,K}(t)\| \leq \|x_{G,K}(t_0)\| e^{-\mu(t-t_0)}, \quad \forall t \geq t_0.$$  

Under the assumption of minimal realization, it is well known that exponential stability is equivalent to the stability of the closed-loop $A$-matrix

$$A_{cl} = \begin{bmatrix} A_G + B_G D_K C_G & B_K C_K \\ B_K C_G & A_K \end{bmatrix},$$

i.e., all its eigenvalues have negative real part. From now on we identify the stability of $\Sigma_1$ with that of matrix $A_{cl}$. Clearly, that $A_{cl}$ is stable is also equivalent to that system $\Sigma_1$ is input-output stable in the sense of $L_p$ induced norm for every $p \in [1, \infty]$. A parallel theory exists for discrete-time systems.

Now digitize the analog system $\Sigma_1$ by replacing $K$ with $HK_{gbt}S$, where $H$ is a zero-order holder, $K_{gbt}$ is an approximation of $K$ via GBT for some $\alpha \in (-\infty, \infty)$ and $S$ is an ideal sampler. Denote the resulting sampled-data system by $\Sigma_2$ as shown in Fig. 3. In what follows we will investigate stability of system $\Sigma_2$. To do so, stability of a certain discrete-time system should be first discussed. Let $G_{zoh} = SGH$, i.e., $G_{zoh}$ is the zero-order hold equivalent of $G$. Then the closed-loop $A$-matrix of the discrete-time system consisting of $G_{zoh}$ and $K_{gbt}$ is given by

$$A_{cl,d} = \begin{bmatrix} A_{Gzoh} + B_{Gzoh} D_{K_{gbt}} C_G & B_{Gzoh} C_{K_{gbt}} \\ B_{K_{gbt}} C_G & A_{K_{gbt}} \end{bmatrix}.$$  

For convenience, denote this discrete-time system by $\Sigma_{2d}$.

The following result is used in proving the stability of $\Sigma_{2d}$:

**Proposition 1:** Let $A \in C^{n \times n}$ and $\epsilon > 0$ be given. Then there is a consistent matrix norm $\|\cdot\|_{sp}$ such that

$$\|I + hA\|_{sp} \leq \rho(I + hA) + \epsilon h,$$  

where $h$ is an arbitrary positive real number.

**Proof.** This result is a variation of a result in [11] (pp. 297, Chapter 5). By the Schur triangularization theorem (Theorem 2.3.1 in [11]), there is a unitary matrix $U$ such that, for every initial time $t\geq t_0$ and initial state $x_{G,K}(t_0)$,

$$\|x_{G,K}(t)\| \leq \|x_{G,K}(t_0)\| e^{-\mu(t-t_0)}, \quad \forall t \geq t_0.$$  

Under the assumption of minimal realization, it is well known that exponential stability is equivalent to the stability of the closed-loop $A$-matrix

$$A_{cl} = \begin{bmatrix} A_G + B_G D_K C_G & B_K C_K \\ B_K C_G & A_K \end{bmatrix},$$

Thus, for $t > 0$ large enough, the sum of all the absolute values of the off-diagonal entries of $Q_t \Xi Q_t^{-1}$ is less than $\epsilon$. Consequently,

$$\|I + hQ_t \Xi Q_t^{-1}\|_1 \leq \rho(I + hA) + \epsilon h$$  

for any $h > 0$. In view of this, define a matrix norm $\|\cdot\|_{sp}$ by

$$\|V\|_{sp} := \|Q_t UVU^* Q_t^{-1}\|_1 = \|((Q_t)^* V (Q_t)^{-1})_1.$$
for any \( V \in \mathbb{C}^{n \times n} \). Then, it is easy to show that
\[
\| I + hA \|_{sp} = \| I + hQ_tU^* \Xi U^* Q_t^{-1} \|_1 = \| I + hQ_t \Xi_t^{-1} \|_1 \leq \rho (I + hA) + \epsilon h.
\]

Moreover, note that for any two matrices \( V_1, V_2 \in \mathbb{C}^{n \times n} \),
\[
\| V_1 V_2 \|_{sp} = \| (Q_t U) V_1 (Q_t U)^{-1} \|_1.
\]
\[
\leq \| (Q_t U) V_1 (Q_t U)^{-1} \|_1 \| (Q_t U) V_2 (Q_t U)^{-1} \|_1 \leq \| V_1 \|_{sp} \| V_2 \|_{sp}.
\]

Consequently, \( \| \cdot \|_{sp} \) is a consistent matrix norm. Define
\[
\begin{align*}
\Gamma &= \min \{ |\text{Re}(\lambda)| : \lambda \text{ is an eigenvalue of } A_{cl} \}, \\
\delta_2 &= \min \left\{ \frac{-a_i}{a_i^2 + b_i^2} \frac{3|b_i| \Gamma}{4(a_i^2 + b_i^2)^{3/2}} \right\}, \\
\Lambda &= \max \{ |\text{Re}(\lambda)| : \lambda \text{ is an eigenvalue of } A_{cl} \}, \\
\delta_3 &= \min \left\{ \delta_2, \frac{1}{\Lambda} \right\}, \\
\Delta(h, \alpha) &= \frac{1}{h^2} (A_{cl,d} - I - hA_{cl}), \\
\eta(\alpha) &= \max_{h \in (0, \delta_3)} \| \Delta(h, \alpha) \|_{sp}, \\
\alpha^* &= \min_{\alpha} \eta(\alpha) = \min_{\alpha \in \mathbb{R}} \max_{h \in (0, \delta_3)} \| \Delta(h, \alpha) \|_{sp}.
\end{align*}
\]

Denote
\[
\eta^* = \eta(\alpha^*).
\]

And define
\[
\delta = \min \left\{ \frac{\Gamma}{4 \eta^*}, \delta_3 \right\}.
\]

We have the following result concerning stability of \( \Sigma_{2d} \):

**Lemma 1:** Suppose that the original continuous-time system \( \Sigma_1 \) is stable. Then for every \( h \in (0, \delta) \), there exists a real number \( \alpha \) such that the closed-loop system \( \Sigma_{2d} \) is stable too, where the positive scalar \( \delta \) is given in Eq. (11).

The proof of Lemma 1 is omitted due to page limitation. \[ \tag{11} \]

**Remark 2:** The function \( \eta(\alpha) \) in Eq. (10) should be minimized in order to get a big \( \delta \) in Eq. (11). Because only two variables, \( \alpha \) and \( h \) are involved, this optimization problem can be solved very easily.

Armed with Lemma 1, we are ready to establish a result concerning the upper bound of the sampling period for the sampled-data system \( \Sigma_2 \). Notice that \( \Sigma_2 \) is a hybrid system, so the first problem is to define its state. Let \( L^* \) be the forward shift operator on \( x_{\text{Kght}} \), where \( x_{\text{Kght}} \) is the state of the discrete-time system \( K_{\text{ght}} \). Define \( x_{\text{hybrid}} = [x_{\text{Kght}}^T, (HL^* x_{\text{Kght}})^T]^T \). Then according to [12], \( x_{\text{hybrid}} \) turns out to be a suitable candidate for the state of the hybrid system \( \Sigma_2 \). Furthermore, the following result holds.

**Theorem 1:** Suppose that the original continuous-time system \( \Sigma_1 \) is stable. Then for every \( h \in (0, \delta) \), there exists a real number \( \alpha \) such that the closed-loop system \( \Sigma_2 \) is stable too, where the positive scalar \( \delta \) is given in Eq. (11).

Theorem 1 is an immediate consequence of Lemma 1 proved above and Theorem 4 in [12].

**IV. STEP TRACKING**

This section focuses on the following problem: Assume that an analog system is step-tracking, will step tracking be preserved when the controller is digitized via the generalized bilinear transformation (GBT)?

Consider the analog system \( \Sigma_3 \) as shown in Fig. 4, where \( r \) is a step input and the transfer functions of the plant \( G \) and the controller \( K \) are given by Eq. (6). Discretize \( K \) using GBT yields a sampled-data system as shown in Fig. 5.

![Fig. 4. Unity-feedback analog system \( \Sigma_3 \)](image)

![Fig. 5. Sampled-data system \( \Sigma_4 \) via GBT](image)

The following result asserts that step tracking is preserved.

**Theorem 2:** Assume that system \( \Sigma_3 \) is stable and step-tracking. Then system \( \Sigma_4 \) is step-tracking if it is stable.

**Proof.** Under the preceding hypothesis, the transfer function from \( r \) to \( e \) in Fig. 4
\[
\begin{bmatrix}
A & B \\
C & D \\
\end{bmatrix} := \begin{bmatrix}
A_G - B_G D_K C_G & B_G C_K \\
-B_K C_G & A_K \\
-C_G & 0 \\
\end{bmatrix} \begin{bmatrix}
B_G D_K \\
B_K \\
I \\
\end{bmatrix}
\]

is stable. Hence, the DC gain from \( r \) to \( e \) is finite and given by
\[
D - CA^{-1}B.
\]

Let \( \rho = Sr \) and \( \varepsilon = Se \). Then the transfer function from \( \rho \) to \( \varepsilon \) is given by the transfer function
\[
\begin{bmatrix}
A & B \\
C & D \\
\end{bmatrix},
\]
where
\[
\begin{align*}
A &= \begin{bmatrix}
A_G - B_G D_K C_G & B_G C_K \\
-B_K C_G & A_K \\
-C_G & 0 \\
\end{bmatrix}, \\
B &= \begin{bmatrix}
B_G D_K \\
B_K \\
\end{bmatrix}, \\
C &= \begin{bmatrix}
-C_G & 0 \\
\end{bmatrix}, \\
D &= I
\end{align*}
\]
Assume that $\Sigma_4$ is stable. Then according to Proposition 1 and Theorem 1, $A$ is stable. So the DC gain from $\rho$ to $\varepsilon$ is stable. Since $r$ is a step input, to show that system $\Sigma_4$ is step-tracking, it suffices to show that

$$D + C(I - A)B = D - CA^{-1}B.$$  

First we prove the case of $\alpha \neq 0$. Define

$$f_t(s) := \int_0^t e^{\alpha s} \, d\tau = \frac{e^{\alpha t} - 1}{\alpha}.$$  

Then

$$f_h(A_G)A_G = A_{Gzhoh} - I, \quad f_h(A_G)B_G = B_{Gzhoh}.$$  

Let $\Pi = \alpha h (I - \alpha h A_G)^{-1}$. Define

$$T_1 = \left[ \begin{array}{c} f_h(A_G) \\ I \end{array} \right] \Pi, \quad T_2 = \left[ \begin{array}{cc} I & 0 \\ 0 & \alpha I \end{array} \right].$$  

It is easy to show that

$$T_1 A T_2 = \left[ \begin{array}{c} f_h(A_G) \\ I \end{array} \right] \Pi \times \left[ \begin{array}{cc} A_G - B_G D_K C_G & B_G C_K \\ -B_K C_G & A_K \end{array} \right] \left[ \begin{array}{cc} I & 0 \\ 0 & \frac{1}{\alpha} I \end{array} \right] = \left[ \begin{array}{cc} \frac{1}{\alpha} I & 0 \\ 0 & I \end{array} \right] (A - I) \left[ \begin{array}{cc} \alpha I & 0 \\ 0 & I \end{array} \right].$$  

Hence

$$[\alpha I \\ 0 \quad 0 \quad I] T_1 A T_2 = [\alpha I \\ 0 \quad 0 \quad I] = A - I.$$  

Similarly,

$$[I \\ 0 \quad \frac{1}{\alpha} I] T_2 B = B, \quad C T_2 = \left[ \begin{array}{ccc} I & 0 & 0 \\ 0 & 0 & \alpha I \end{array} \right] = C, \quad D = D.$$  

As a consequence,

$$D + C(I - A)^{-1}B = D - CT_2 \left[ \begin{array}{ccc} I & 0 & 0 \\ 0 & 0 & \alpha I \end{array} \right] T_2^{-1} A^{-1} T_1^{-1} \left[ \begin{array}{cc} \alpha I & 0 \\ 0 & I \end{array} \right] T_1 B = D - CA^{-1}B.$$  

Next we prove the case of $\alpha = 0$. Define

$$\Pi = h I, \quad T_1 = \left[ \begin{array}{c} f_h(A_G) \\ I \end{array} \right] \Pi, \quad T_2 = I,$$  

it readily follows that

$$T_1 A T_2 = A - I, \quad T_2 B = B, \quad C T_2 = C, \quad D = D.$$  

Consequently,

$$D + C(I - A)^{-1}B = D - CA^{-1}B.$$  

The result is proven.

According to Theorem 2, the sampled-data system $\Sigma_4$ in Fig. 5 may be step-tracking for many values of $\alpha$. Not surprisingly, different choices of $\alpha$ may give rise to vastly different transient response, as shown by the following example. Therefore, GBT may improve considerably the step-tracking performance of sampled-data systems when the performance is optimized over $\alpha$.

**Example 1:** Assume that $G$ in Fig. 4 is given by

$$G(s) = \frac{1}{(10s + 1)(25s + 1)}.$$

Moreover, suppose

$$K(s) = \frac{513.04775s^2 + 7.9128s + 1.3475}{s^2 + 1.7231s}$$

is designed to achieve step tracking. Now we digitize system $\Sigma_3$ using GBT. Fix the sampling period $h = 1.2$. Simulation results are given in Fig. 6. Clearly, the output of the sampled-data system obtained via the (traditional) bilinear transformation has much larger overshoot as well as longer settling time than that obtain via GBT with $\alpha = 0.1$.

![Step Tracking](image)

**V. AN APPLICATION**

**Example 2:** Consider the analog system as shown in Fig. 4, where $r$ is a step input and the transfer functions of the plant $G$ and the controller $K$ are given by Eq. (12):

$$G(s) = \frac{10}{s(s + 1)}, \quad K(s) = \frac{0.416s + 1}{0.139s + 1}$$

Discretize $K$ using GBT yields a sampled-data system as shown in Fig. 5. This example is first studied in [10]. It is reported that common digitization methods yield either non-stabilizing controllers or systems with very poor closed-loop performance. A digital re-design approached in [5] which can produce controllers guaranteeing satisfactory closed-loop
performance even when the sampling period $h$ is large. In this part, we will compare GBT and that method to demonstrate the strength of GBT.

We compare these two methods by studying unit step response of close-loop systems for sampling periods $h = 0.157$ (Fig. 7), $h = 0.314$ (Fig. 8), and $h = 0.42$ (Fig. 9), respectively.

It is shown that, when $h > 0.42$, the plant is not stabilizable via controller designed using method in [5]. However, for $h \in (0, 0.578)$, it can be stabilized by controllers designed via GBT. Moreover, controllers obtained based on the method in [5] are of second order, while those via GBT are of first order. Clearly, GBT yields better closed-loop performance even for large sampling periods.

VI. CONCLUSION

In this paper we have proposed a new controller discretization approach, namely, the generalized bilinear transformation (GBT). GBT provides a class of digital approximations of an analog controller, thus optimal discretization is rendered possible. We have studied the stability as well as step tracking of sampled-data systems obtained via GBT. Numerical examples demonstrate the effectiveness of the controller re-design approach.

REFERENCES


