Single Photon Inverting Pulse for an Atom In a Cavity

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Abstract—This paper considers the model where a two-level atom is placed in an open cavity. We show that the atom-cavity system can be taken as a linear quantum system under the driving of single photon state. We prove that the single photon input to the cavity could coherently invert the atomic state from the ground state to the excited state. We derive the exact expressions of the single photon inverting pulses when the open cavity is open to single-channel and two-channel input.

I. INTRODUCTION

Quantum systems could be manipulated either by classical control or quantum coherent control [14], [4], [1], [17]. Classical control often uses classical fields that can be understood as complex-valued variables. In contrast, quantum coherent control uses quantum signals. Coherent quantum signals, which could be generated as the output state of a quantum system, can be used as the input to another quantum system. This scheme may find application in the design of a feedback loop for feedback control, or in the construction of a large-scale quantum system by interconnection of small quantum systems. In these applications, the quantum control fields can no longer be modelled as classical complex-valued variables, but need to be characterized as operators on Hilbert spaces.

In this paper, we consider the input to the quantum system as a continuous-mode single photon state. A continuous-mode single photon state is a quantum superposition of single photon excitations at different times, and the probability distribution of the single photon excitations can be characterized by a pulse shape of finite width. Single photon state is of critical importance to quantum information processing [9], and recently its use in the engineering of quantum switch and transistor has been considered [2], [3]. This trend motivates us to study the interaction between a single photon input and the quantum system. The response of quantum linear system to single photon input has been investigated in [16], [15], and the interaction between a single photon input and a two-level quantum system has been studied as well [5], [12].

We aim to derive the single photon inverting pulse that can coherently deliver the single photon it carries into an atom, which result in a full inversion of the atomic state. This kind of pulses have potential applications in quantum memory, as well as coherent control gated by photons. To do this, we observe that the atom can be taken as a linear system when the input is a single photon state [10]. Therefore, we can make use of a linear transfer function approach which is similar to the one derived in [16], [15]. In this paper, we consider an extended model, where the atom is placed in a cavity. Thus the atom interacts with the input field indirectly, mediated by the cavity. We will prove that the atom-cavity system can also be taken exactly as a linear system when subjected to single photon input. Consequently, we can build upon the approach from [10] and zero-dynamics principle [15] to derive the inverting pulses that can fully excite the atom within the cavity.

In Section II-V, we give the introduction of quantum linear and two-level systems and the existing results on the single photon inverting pulse for an atom. The single photon inverting pulses are derived in VI for the atom placed in a cavity. Conclusion is put in the last section.

II. QUANTUM LINEAR SYSTEMS

Quantum system is defined on a Hilbert space ${{\mathcal{H}}}$. The quantum system must be open to the input for the coherent control. The input could be defined on a Fock space ${{\mathcal{H}}_d}$ over $L^2({\mathbb{R}_+},dt)$. The system evolution is described in the Heisenberg picture by the time evolution of observables (self-adjoint operators) defined on the system space. To be more precise, the observable $X$ evolves as $X(t) = U(t)^\dagger (X \otimes I) U(t)$, where $X(t_0) = X$ is defined on ${{\mathcal{H}}}$. $U(t)$ is the unitary evolution operator of the total system defined on ${{\mathcal{H}}} \otimes {{\mathcal{H}}_d}$. The dynamical equation for $X(t)$ can be expressed using the quantum stochastic differential equation [7], [11]

$$
dX(t) = (-i[X,H_0] + {\mathcal{L}(X)})dt + \sum_{k=1}^{K} [L_k^\dagger(X(t),L_k(t))]dB_k(t),$$

with

$$
{\mathcal{L}(X)} = \sum_{k=1}^{K} L_k^\dagger X L_k - \frac{1}{2} L_k^\dagger L_k X - \frac{1}{2} X L_k^\dagger L_k,$$

where $H_0$ is the Hamiltonian of the system, $L_k$ describes the coupling between the system and the $k$-th of the total $K$ input Bosonic fields. $B_k(t)$ and $B_k^\dagger(t)$ are the annihilation and creation processes defined on ${{\mathcal{H}}_d}$, with $dB_k(t)$ and $dB_k^\dagger(t)$ being the Ito increments. Equation (1) defines a Markov process. In the common setting of quantum optics, we can write $B_k(t) = \int_0^t b_k(s)ds$ and $B_k^\dagger(t) = \int_0^t b_k^\dagger(s)ds$, with $b_k(s)$ and $b_k^\dagger(s)$ being the creation and annihilation operators of the $k$-th mode at time $s$. These operators satisfy the condition

$$[b_k(t), b_k^\dagger(s)] = \delta(t-s).$$

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A single-mode optical cavity is the simplest quantum linear system [13]. The creation and annihilation operators for the single mode of the cavity is denoted as $a$ and $a^\dagger$ satisfying $[a, a^\dagger] = 1$. The system Hamiltonian of a single-mode cavity can be written as $H_0 = \omega_a a^\dagger a$, where $\omega_a$ is the characteristic frequency of the cavity. Often we can let $\omega_a = 0$ by proper detuning. We call it single-channel input if there is only one input field, e.g., $K = 1$. Thus we can denote the creation and annihilation operators of the input field as $b^\dagger(t)$ and $b(t)$ in single-channel case. According to (1), the Heisenberg-picture evolution of $a(t)$ in response to a single-channel input is given by

$$
\dot{a}(t) = -\frac{\kappa}{2} a(t) + \sqrt{\kappa} b(t),
$$

by assuming the coupling $L = \sqrt{\kappa} a$. $\kappa$ is the coupling strength. Plus, taking $b(t)$ as the input we have the input-output relation of the cavity as

$$
b_{\text{out}}(t) = \sqrt{\kappa} a(t) + b(t),
$$

where $b_{\text{out}}(t)$ is defined by $b_{\text{out}}(t) = U(t)^\dagger b(t)U(t)$. The equations (4) – (5) are analogous to the classical dynamical and input-output equations for linear systems.

III. QUANTUM TWO-LEVEL SYSTEM

Quantum two-level system, which is also known as qubit, is defined using two basis states $|0\rangle$ and $|1\rangle$. $|0\rangle$ is called the ground state and $|1\rangle$ is called the exited state, and these two states can be used to denote the two atomic states of a physical atom. The Pauli operators for a qubit are defined as $\sigma_x = |1\rangle\langle 0| - |0\rangle\langle 1|$, $\sigma_y = i(|0\rangle\langle 1| + |1\rangle\langle 0|)$, $\sigma_z = -i|1\rangle\langle 1| + i|0\rangle\langle 0|$. The raising and lowering operators for the qubit, whose functions are analogous to the creation and annihilation operators of optical fields, are given by $\sigma_+ = |1\rangle\langle 0|$, $\sigma_- = |0\rangle\langle 1|$. Clearly, we have

$$
\sigma_+ |1\rangle = |1\rangle, \quad \sigma_- |0\rangle = |0\rangle.
$$

According to (1), the Heisenberg-picture evolution of $\sigma_-(t)$ in response to a single-channel input $b(t)$ is given by

$$
\dot{\sigma}_-(t) = -\frac{\kappa}{2} \sigma_-(t) + \sqrt{\kappa} \sigma_+ b(t).
$$

The second term on the RHS of (7) is bilinear, which makes the integration of (7) nontrivial in general.

IV. SINGLE PHOTON STATE

A continuous-mode single photon input, which has a finite temporal extent, is a superposition of single photon excitations at different times. It could be defined by [8], [6]

$$
|1_\xi\rangle = B^\dagger(\xi)|0\rangle = \int_0^{\infty} \xi(t)b^\dagger(t)|0\rangle dt,
$$

where $\xi(t)$ represents the pulse shape of the single photon in the time domain. $\int_0^{\infty} \xi(t)^2 dt$ is the probability of finding the photon during $[t, t + dt]$. Clearly, we have the normalization condition $\int_0^{\infty} \xi(t)^2 dt = 1$ for a single photon input. A full inverting single photon pulse will transfer the single photon of input completely to the atom, which will result in an excited state $|1_\xi\rangle$ of the system if its initial state is the ground state.

V. SINGLE PHOTON INVERTING PULSE FOR AN ATOM

If the atom is directly interacting with the field like we discussed in Section III, the system dynamics is governed by a bilinear equation (7). However we have shown in [10] that (7) can be integrated as a linear equation in response to a single photon input. This is possible partly due to the following relation

$$
\langle 0, 0 | \sigma_-(t) = -\langle 0, 0 | \frac{\kappa}{2} \sigma_-(t) + \sqrt{\kappa} \langle 0, 0 | \sigma_+ b(t)
$$

$$
= -\langle 0, 0 | \frac{\kappa}{2} \sigma_-(t) + \sqrt{\kappa} \langle 0, 0 | U^\dagger(t) \sigma(t) U(t) b(t)
$$

$$
= -\langle 0, 0 | \frac{\kappa}{2} \sigma_-(t) - \sqrt{\kappa} \langle 0, 0 | b(t),
$$

where $\langle 0, 0 |$ is the state with the field being vacuum and the atom being at the ground state.

We can prove that the single photon inverting pulse for the atom is analogous to the single photon inverting pulse for a cavity, which is given by

$$
\xi(t) = \begin{cases} 
-\sqrt{\kappa} \exp(\frac{\imath \xi}{2}), & t \leq 0, \\
0, & t > 0,
\end{cases}
$$

using the zero-dynamics principle [15], [10]. At $t = 0$, the single photon input is completely absorbed into the atom.

VI. SINGLE PHOTON INVERTING PULSE FOR AN ATOM PLACED IN A CAVITY

In this section, we consider the model where the two-level atom is placed inside a single-mode cavity. The cavity is coupled to the single photon input, while the atom is coupled to the cavity through an interaction. The atom itself is not coupled to the single photon input, but we still want to derive the single photon inverting pulses that can transfer a single photon to the atom through the cavity. We will consider two cases, namely, single-channel and two-channel input to the cavity, which correspond to one-sided and two-sided optical cavity [13], respectively.

A. One-channel

A two-level atom is placed inside a cavity. Only the cavity is open to input and output channels. The internal Hamiltonian of the cavity-atom system is

$$
H_0 = \frac{\omega_a}{2} \sigma_z + \omega_a a^\dagger a + ig(a^\dagger \sigma_+ - \sigma_- a).
$$

Here $\omega_a$ is the frequency of the cavity, and $\omega_a$ is the frequency of the cavity. $\omega_a$ also indicates the energy difference between the two levels of the atom. $g$ is the interaction Hamiltonian between the cavity and the atom. $g$ is the coupling strength. The cavity is coupled to a single-channel input via a coupling operator $L = \sqrt{\kappa} a$. According to (1), the Heisenberg-picture evolution of $\sigma_+(t)$ and $a^\dagger(t)$ is given by

$$
\begin{pmatrix}
\dot{\sigma}_+(t) \\
\dot{a}^\dagger(t)
\end{pmatrix} = \tilde{A}
\begin{pmatrix}
\sigma_+(t) \\
\sigma_-^\dagger(t)
\end{pmatrix} + C b^\dagger(t),
$$

with

$$
\tilde{A} = \begin{pmatrix}
\imath \omega_a & g \\
\frac{\kappa}{2} + \imath \omega_a & -\frac{\kappa}{2}
\end{pmatrix}, \quad C = \begin{pmatrix}
0 \\
-\sqrt{\kappa}
\end{pmatrix}.
$$
Here $\tilde{A}$ is not a constant matrix in general due to the Heisenberg-picture operator $\sigma_z(t)$.

Now suppose the initial state is $|0,0\rangle_0$. $|0,0\rangle$ is the ground state of the atom-cavity system. The first component of $|0,0\rangle$ is for the state of the atom, while the second component of $|0,0\rangle$ is for the state of the cavity. If the single-photon input is fully absorbed into the atom, then the atom-cavity state is $|1,0\rangle$. Eq. (12) acting on this initial state yields the following dynamical equation

$$\begin{pmatrix} \sigma_+(t) \\ a^+(t) \end{pmatrix} |0,0\rangle_0 = [A \begin{pmatrix} \sigma_+(t) \\ a^+(t) \end{pmatrix} + C b^+(t)] |0,0\rangle_0,$$

where we obtain a constant coefficient matrix

$$A = \begin{pmatrix} i\omega_a & -g \\ g & -\frac{\kappa}{2} + i\omega_c \end{pmatrix}.$$  (15)

In order to obtain the linear dynamical equation (14), we have made use of the following relation

$$\sigma_z(t)|0,0\rangle_0 = U^+(t)\sigma_z U(t)|0,0\rangle_0 = -U^+(t)|0,0\rangle_0 = -|0,0\rangle_0.$$  (16)

The linear dynamical equation (14) can be solved as

$$\begin{pmatrix} \sigma_+(t_1) \\ a^+(t_1) \end{pmatrix} |0,0\rangle_0 = e^{A(t_1-t_0)} \begin{pmatrix} \sigma_+(t_0) \\ a^+(t_0) \end{pmatrix} + \int_{t_0}^{t_1} e^{A(t_1-t)} C b^+(t)dt |0,0\rangle_0.$$  (17)

Letting $t_0 \to -\infty$ we get

$$\begin{pmatrix} \sigma_+(t_1) \\ a^+(t_1) \end{pmatrix} |0,0\rangle_0 = \int_{-\infty}^{t_1} e^{A(t_1-t)} C b^+(t)dt |0,0\rangle_0.$$  (18)

Here we have used the property that $A$ is Hurwitz (See Appendix for details). Now we define the exponentially rising pulses as did in [15]

$$\begin{pmatrix} \tilde{v}_1(t) \\ \tilde{v}_2(t) \end{pmatrix} = e^{A(t_1-t)} C,$$

and

$$\begin{pmatrix} v_1(t) \\ v_2(t) \end{pmatrix} = \begin{pmatrix} \tilde{v}_1(t) \Theta(t_1-t) \\ \tilde{v}_2(t) \Theta(t_1-t) \end{pmatrix}.$$  (20)

Then Eq. (18) can be rewritten as

$$\begin{pmatrix} \sigma_+(t_1) \\ a^+(t_1) \end{pmatrix} |0,0\rangle_0 = \int_{-\infty}^{t_1} \begin{pmatrix} v_1(t) \\ v_2(t) \end{pmatrix} b^+(t)dt |0,0\rangle_0 = \begin{pmatrix} B^+(v_1) \\ B^+(v_2) \end{pmatrix} |0,0\rangle_0.$$  (21)

It is easy to verify that $B^+(v_1)$ and $B^+(v_2)$ both could generate a single photon input, with the pulse shape given by $v_1(t)$ and $v_2(t)$, respectively. In particular, we have

$$\sigma_+(t_1)|0,0\rangle_0 = B^+(v_1)|0,0\rangle_0.$$  (22)

If the pulse shape of the single photon input is given by $v_1(t)$, then the final state of the system at $t = t_1$ is thus calculated by

$$|\Psi(t_1)\rangle = U(0,\infty, t_1)|0,0\rangle|1_{v_1}\rangle = U(0,\infty, t_1)B^+(v_1)|0,0\rangle|0\rangle = U(0,\infty, t_1)\sigma_+(t_1)|0,0\rangle|0\rangle = U(0,\infty, t_1)\sigma_+(t_1)U^+(0,\infty, t_1)|0,0\rangle|0\rangle = U(0,\infty, t_1)\sigma_+(t_1)U^+(0,\infty, t_1)|0,0\rangle|0\rangle.$$

For simplicity, we let $t_1 = 0$. By (19)-(20) and (23), the shape of the single photon inverting pulse for the atom is given by the first component of the vector

$$e^{-\lambda^+_t C}, \quad t \leq 0.$$  (24)

According to the Cayley-Hamilton theorem, the exponential matrix $e^{Mt}$ can be written as

$$e^{Mt} = s_0(t)I + s_1(t)M.$$  (25)

Furthermore, the coefficients $s_0(t)$ and $s_1(t)$ can be expressed as

$$s_0(t) = \frac{\alpha \exp(\beta t) - \beta \exp(\alpha t)}{\alpha - \beta},$$

$$s_1(t) = \frac{\exp(\alpha t) - \exp(\beta t)}{\alpha - \beta},$$

where $\alpha, \beta$ are eigenvalues of $M$ and $\alpha \neq \beta$. Using these relations, the inverting pulse is calculated to be

$$v_1(t) = \frac{\sqrt{\kappa g}}{\lambda^+_t - \lambda^-_t} (e^{\lambda^+_t t} - e^{\lambda^-_t t}), \quad t \leq 0$$

for $\lambda^+_t \neq \lambda^-_t$, where $\lambda^+_t$ are eigenvalues of $-A$. More explicitly, we have the following theorem:

**Theorem 1:** The single photon inverting pulse for single-channel case is given by

$$v_1(t) = \frac{\sqrt{\kappa g}}{\sqrt{[-\frac{\kappa}{2} + i(\omega_a - \omega_c)]^2 - 4g^2}} \left[ \exp\left(\frac{\sqrt{\kappa g}}{\sqrt{[-\frac{\kappa}{2} + i(\omega_a - \omega_c)]^2 - 4g^2}} t \right) - \exp(\frac{\sqrt{\kappa g}}{\sqrt{[-\frac{\kappa}{2} + i(\omega_a - \omega_c)]^2 - 4g^2}} t) \right].$$

**Proof:** Just need to calculate the eigenvalues of $-A$ explicitly.

It is easy to see that this inverting pulse is always valid if the atom and the cavity are out of resonance, i.e. $\omega_a \neq \omega_c$. In this case we always have $\lambda^+_t \neq \lambda^-_t$.

In particular, if the atom is largely detuned from the cavity, i.e. $g, \kappa \ll |\omega_a - \omega_c|$, we have

$$v_1(t) \approx \frac{\sqrt{\kappa g}}{\sqrt{[-\frac{\kappa}{2} + i(\omega_a - \omega_c)]^2 - 4g^2}} \left[ \exp(-i\omega_a t) - \exp(-i\omega_c t) \right], \quad t \leq 0.$$  (29)
When the cavity and the atom are in resonance, i.e., $\omega_a = \omega_c$, Eq. (28) can be simplified as

$$v_1(t) = -\frac{\sqrt{\kappa}}{\sqrt{\kappa^2 - 4g^2}}\left[\exp\left(\frac{\kappa}{2} - 2i\omega_c + \frac{\kappa^2}{4} - 4g^2\right)t\right] - \exp\left(-\frac{\kappa}{2} - 2i\omega_c - \frac{\kappa^2}{4} - 4g^2\right)t], \ t \leq 0. \quad (30)$$

Obviously, if $\kappa^2 - 4g^2 = 0$, (30) becomes singular and hence is not feasible.

**B. Two-channel**

Similarly, if the atom-cavity system couples to the input via two channels as $L = (\sqrt{K}a^\dagger \sqrt{K}a)^T$, then we need to solve the following dynamical equation

$$\begin{pmatrix}
\sigma_+(t) \\
a^\dagger(t)
\end{pmatrix} = [A \begin{pmatrix}
\sigma_+(t) \\
a^\dagger(t)
\end{pmatrix} + C \begin{pmatrix}
b_1^\dagger(t) \\
b_2^\dagger(t)
\end{pmatrix}]|0,0\rangle|00\rangle, \quad (31)
$$

with

$$A = \begin{pmatrix}
\frac{i\omega_e}{g} - \frac{\kappa_1 + g}{2\sqrt{\kappa_1}} - i\omega_c \\
0
\end{pmatrix},$$

$$C = \begin{pmatrix}
0 \\
0 - \sqrt{\kappa_1} - \sqrt{\kappa_2}
\end{pmatrix}. \quad (32)$$

It is easy to verify that $A$ is Hurwitz. $|00\rangle$ is the input state with both fields being vacuum. The single photon inverting pulse for the atom is still given by the first component of the vector

$$e^{-At}C \begin{pmatrix}
b_1^\dagger(t) \\
b_2^\dagger(t)
\end{pmatrix}, \ t \leq 0. \quad (33)$$

Simple calculation shows that the inverting pulse distributes on both channels. More explicitly, we have

**Theorem 2:** The inverting pulse for two-channel case is

$$|\xi_1\xi_2\rangle = \int_0^\infty \left[\xi_1(t)b_1^\dagger(t)dt + \xi_2(t)b_2^\dagger(t)dt\right]|00\rangle, \ t \leq 0, \quad (34)$$

where $\xi_j(t), j = 1,2$ are expressed as

$$\begin{align*}
\xi_j(t) &= -\frac{\sqrt{\kappa}}{2\sqrt{\kappa^2 - 4g^2}}\left[\exp\left(\frac{\kappa_1 + g}{2} - i(\omega_e + \omega_a)t + \frac{\kappa_1 + g}{2}i(\omega_e - \omega_a)t\right)\right] \\
&\quad - \exp\left(-\frac{\kappa_1 + g}{2} - i(\omega_e + \omega_a)t - \frac{\kappa_1 + g}{2}i(\omega_e - \omega_a)t\right)].
\end{align*} \quad (35)$$

Since $\kappa_1, g > 0$, we cannot make $\xi_j(t) = 0$ for either $j = 1$ or $j = 2$. This is because the only way we can let $\xi_j(t) = 0$ is by establishing

$$\exp\left(\frac{\kappa_1 + g}{2} - i(\omega_e + \omega_a)t + \frac{\kappa_1 + g}{2}i(\omega_e - \omega_a)t\right) - \exp\left(-\frac{\kappa_1 + g}{2} - i(\omega_e + \omega_a)t - \frac{\kappa_1 + g}{2}i(\omega_e - \omega_a)t\right) = 0, \quad (36)$$

which leads to

$$[-\frac{\kappa_1 + g}{2} + i(\omega_e - \omega_a)]^2 - 4g^2 = 0. \quad (37)$$

Obviously, the condition (37) will not generate a feasible pulse shape for $\xi_j(t)$.

We have proven that it is impossible to use a uni-direction single photon inverting pulse to fully excite the atom when the atom-cavity system is coupled to two input channels.

**VII. Conclusion**

We have successfully obtained the single photon inverting pulse to store one photon into the two-level atom embedded in a linear cavity. If the cavity has two input channels, the single photon inverting pulse has to be divide into both channels and then interact with the cavity.

**APPENDIX**

The characteristic equation of $A$ is given by

$$\lambda^2 + \left[\frac{\kappa}{2} - i(\omega_e + \omega_a)\right]\lambda - i\omega_e\left(\frac{\kappa}{2} - i\omega_c\right) + g^2 = 0. \quad (38)$$

Then the eigenvalues of $A$ are explicitly calculated to be

$$\lambda_{\pm} = \frac{-\frac{\kappa}{2} + i(\omega_e + \omega_a) \pm \sqrt{\left(\frac{\kappa}{2} - i(\omega_e - \omega_a)\right)^2 - 4g^2}}{2}. \quad (39)$$

We let $e + if = \sqrt{\left(\frac{\kappa}{2} - i(\omega_e - \omega_a)\right)^2 - 4g^2}$ with $e, f$ being real numbers. We have

$$e^2 - f^2 = \left(\frac{\kappa}{2}\right)^2 - (\omega_e - \omega_a)^2 - 4g^2,$$

$$2ef = -\kappa(\omega_e - \omega_a). \quad (40)$$

So we have

$$e^2 - \frac{(\kappa e)^2}{2}(\omega_e - \omega_a)^2 = \left(\frac{\kappa}{2}\right)^2 - (\omega_e - \omega_a)^2 - 4g^2. \quad (41)$$

Suppose $e \geq \frac{\kappa}{2}$ is satisfied, we have

$$e^2 \leq \left(\frac{\kappa}{2}\right)^2 - 4g^2. \quad (42)$$

We obtain the following inequality using $g^2 > 0$:

$$e^2 < \left(\frac{\kappa}{2}\right)^2, \quad (43)$$

which contradicts $e \geq \frac{\kappa}{2}$. Thus we have proven that any $e$ satisfying $e \geq \frac{\kappa}{2}$ will not lead to a solution to (39). Consequently, the real parts of $\lambda_{\pm}$ are strictly negative, which proves that $A$ is Hurwitz.

**REFERENCES**


