

## CALIBRATING LEAST SQUARES SEMIDEFINITE PROGRAMMING WITH EQUALITY AND INEQUALITY CONSTRAINTS\*

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**Abstract.** In this paper, we consider the least squares semidefinite programming with a large number of equality and inequality constraints. One difficulty in finding an efficient method for solving this problem is due to the presence of the inequality constraints. In this paper, we propose to overcome this difficulty by reformulating the problem as a system of semismooth equations with two level metric projection operators. We then design an inexact smoothing Newton method to solve the resulting semismooth system. At each iteration, we use the BiCGStab iterative solver to obtain an approximate solution to the generated smoothing Newton linear system. Our numerical experiments confirm the high efficiency of the proposed method.

**Key words.** covariance matrix, smoothing Newton method, quadratic convergence

**AMS subject classifications.** 49M29, 90C25, 90C30

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**1. Introduction.** In this paper, we are interested in the following least squares semidefinite programming (LSSDP):

$$(1) \quad \begin{aligned} \min \quad & \frac{1}{2} \|X - C\|^2 \\ \text{s.t.} \quad & \langle A_i, X \rangle = b_i, \quad i = 1, \dots, p, \\ & \langle A_i, X \rangle \geq b_i, \quad i = p + 1, \dots, m, \\ & X \in \mathcal{S}_+^n, \end{aligned}$$

where s.t. stands for subject to,  $\mathcal{S}^n$  and  $\mathcal{S}_+^n$  are, respectively, the space of  $n \times n$  symmetric matrices and the cone of positive semidefinite matrices in  $\mathcal{S}^n$ ,  $\|\cdot\|$  is the Frobenius norm induced by the standard trace inner product  $\langle \cdot, \cdot \rangle$  in  $\mathcal{S}^n$ ,  $C$  and  $A_i$ ,  $i = 1, \dots, m$ , are given matrices in  $\mathcal{S}^n$ , and  $b \in \mathbb{R}^m$ . Mathematically, the LSSDP problem (1) can be equivalently written as

$$(2) \quad \begin{aligned} \min \quad & t \\ \text{s.t.} \quad & \langle A_i, X \rangle = b_i, \quad i = 1, \dots, p, \\ & \langle A_i, X \rangle \geq b_i, \quad i = p + 1, \dots, m, \\ & t \geq \|X - C\|, \\ & X \in \mathcal{S}_+^n. \end{aligned}$$

Problem (2) is a linear optimization problem with linear equality/inequality, the second order cone, and the positive semidefinite cone constraints. This suggests that one

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may then use well developed and publicly available software, based on interior point methods (IPMs), such as SeDuMi [45], SDPT3 [48], and a few others to solve (2) and thus the LSSDP problem (1), directly. This is indeed feasible on a Pentium IV PC (the computing machine that we will use in our numerical experiments) as long as  $n$  is small (say, 80 at most) and  $m$  is not too large (say, 5,000). The reason is that at each iteration these solvers require us to formulate and solve a linear system with a dense Schur complement matrix (for example, see [7]) of the size  $(m+1+\bar{n}) \times (m+1+\bar{n})$ , where  $\bar{n} := \frac{1}{2}n(n+1)$ .

Realizing the difficulties in using IPMs to solve the LSSDP problem, in two recent papers, Malick [29] and Boyd and Xiao [10] proposed, respectively, to apply classical quasi-Newton methods (in particular, the BFGS method) and the projected gradient method to the Lagrangian dual of problem (1) as the objective function in the corresponding Lagrangian dual (dual in short) problem is continuously differentiable. Unlike the IPMs, these two dual based approaches are relatively inexpensive at each iteration as the dual problem is of dimension  $m$  only. The overall numerical performance of these two approaches varies from problem to problem. They may take dozens of iterations for some testing examples and several hundreds or thousands for some others.

Historically, the Lagrangian dual based approach has been known to the optimization and approximation theory communities for a long time and has been discussed extensively during the last three decades. Rockafellar's monograph [42] is an excellent source to start with. The LSSDP problem (1) is a special case of the best approximation problem [15],

$$(3) \quad \begin{aligned} \min \quad & \frac{1}{2} \|x - c\|^2 \\ \text{s.t.} \quad & \mathcal{A}x \in b + \mathcal{Q}, \\ & x \in K, \end{aligned}$$

where  $\mathcal{X}$  is a real Hilbert space equipped with a scalar product  $\langle \cdot, \cdot \rangle$  and its induced norm  $\|\cdot\|$ ,  $\mathcal{A} : \mathcal{X} \rightarrow \mathbb{R}^m$  is a bounded linear operator,  $\mathcal{Q} = \{0\}^p \times \mathbb{R}_+^q$  is a polyhedral convex cone,  $1 \leq p \leq m$ ,  $q = m - p$ , and  $K$  is a closed convex cone in  $\mathcal{X}$ . See Deutsch [15] for a comprehensive treatment on the best approximation problems in inner product spaces. The Lagrangian dual (see, e.g., Borwein and Lewis [9]) of the best approximation problem (3) takes the form

$$(4) \quad \begin{aligned} \max \quad & -\frac{1}{2} \|\Pi_K(c + \mathcal{A}^*y)\|^2 + \langle b, y \rangle + \frac{1}{2} \|c\|^2 \\ \text{s.t.} \quad & y \in \mathcal{Q}^*, \end{aligned}$$

where  $\mathcal{A}^* : \mathbb{R}^m \rightarrow \mathcal{X}$  is the adjoint of  $\mathcal{A}$ ,  $\mathcal{Q}^* = \mathbb{R}^p \times \mathbb{R}_+^q$  is the dual cone of  $\mathcal{Q}$ , and for any  $x \in \mathcal{X}$ ,  $\Pi_K(x)$  is the metric projection of  $x$  onto  $K$ ; i.e.,  $\Pi_K(x)$  is the unique optimal solution to

$$\begin{aligned} \min \quad & \frac{1}{2} \|u - x\|^2 \\ \text{s.t.} \quad & u \in K. \end{aligned}$$

Define  $\theta : \mathbb{R}^m \rightarrow \mathbb{R}$  by

$$(5) \quad \theta(y) := \frac{1}{2} \|\Pi_K(c + \mathcal{A}^*y)\|^2 - \langle b, y \rangle - \frac{1}{2} \|c\|^2, \quad y \in \mathbb{R}^m.$$

Note that  $-\theta(\cdot)$  is the objective function in problem (4). This definition of  $\theta(\cdot)$  is for the convenience of subsequent discussions. Such a defined function  $\theta(\cdot)$  is a

convex function [42, 9]. From Zarantonello [51], we know that  $\theta(\cdot)$  is continuously differentiable (but not twice continuously differentiable) with

$$\nabla\theta(y) = \mathcal{A}\Pi_K(c + \mathcal{A}^*y) - b, \quad y \in \mathbb{R}^m,$$

and that  $\nabla\theta(\cdot)$  is Lipschitz continuous. Thus, the dual problem (4) turns out to be a smooth convex optimization problem with a simple constraint:

$$(6) \quad \begin{array}{ll} \min & \theta(y) \\ \text{s.t.} & y \in \mathcal{Q}^*. \end{array}$$

Due to its wide applications in interpolation, the approximation theory community paid a lot of attention in the 1980s and 1990s to the best approximation problem (3) when the inequality constraints are absent (i.e.,  $q = 0$ ). See, for example, [4, 17, 26, 30, 31], to name only a few. For a summary, see [32]. In the case when  $q = 0$ , Micchelli and Utreras in their influential paper [31] suggested that a quasi-Newton method be used to solve the dual problem (6), which is an unconstrained convex optimization problem. In [16], Deutsch, Li, and Ward introduced a steepest descent method for solving the same problem. One particular well-studied best approximation problem is the convex best interpolation problem, where  $K = \{x \in L_2[0, 1] \mid x \geq 0 \text{ a.e. on } [0, 1]\}$ . For the dual of the convex best interpolation problem, Newton's method appears to be very efficient [26, 4, 17] despite the fact that  $\theta(\cdot)$  may not be twice differentiable. By using the superlinear (quadratic) convergence theory of Clarke's generalized Jacobian based Newton methods for solving semismooth equations (an important subclass of nonsmooth equations) established by Kummer [27] and Qi and Sun [38], Dontchev, Qi, and Qi clearly explained the performance of Newton's method for solving the dual of the convex best approximation problem in [18, 19].

Since the metric projection operator  $\Pi_{\mathcal{S}_+^n}(\cdot)$  over the cone  $\mathcal{S}_+^n$  has been proven to be strongly semismooth in [46], the effectiveness of Newton's method for the convex best interpolation problem inspired Qi and Sun [34] to study a quadratically convergent inexact semismooth Newton method to solve the following nearest correlation matrix problem under the  $W$ -weight (after an equivalent transformation):

$$(7) \quad \begin{array}{ll} \min & \frac{1}{2}\|X - C\|^2 \\ \text{s.t.} & (W^{-1/2}XW^{-1/2})_{ii} = 1, \quad i = 1, \dots, n, \\ & X \in \mathcal{S}_+^n, \end{array}$$

where the positive definite matrix  $W \in \mathcal{S}^n$  is given as the  $W$ -weight to the problem and  $W^{1/2}$  is the positive square root of  $W$ . The nearest correlation matrix problem (7) mainly comes from the finance and insurance/reinsurance industries [43, 50]. In real-world applications, for instance, when doing scenarios analysis in stress testing, some covariance entries are restricted to stay with their confidence intervals. This naturally leads to a nearest correlation matrix problem of both equality and inequality constraints, which is a special case of the LSSDP (1). See [1, 2] for related finance problems of inequality constraints.

Higham first formulated the problem (7) in [25] and considered applying Dykstra's alternating projection algorithm [20] to solve it. Numerical experiments conducted in [34, 8] demonstrate clearly that Newton's method outperforms the alternating projection method and the BFGS method. In section 5, we can see that Newton's method combined with a conjugate gradient iterative solver is much faster than the alternating

projection method and the BFGS method when  $W$  is a randomly generated diagonal matrix.

The inexact semismooth Newton method introduced in [34] can certainly be used to solve, at least conceptually, the LSSDP problem (1) in the absence of inequality constraints (i.e.,  $m = p$ ), but it is not applicable to the cases with inequality constraints (i.e.,  $m > p$ ). The reason is that the globally convergent method in [34] can only be applied to an unconstrained convex problem, while its corresponding dual problem (6) with  $K = \mathcal{S}_+^n$  is no longer an unconstrained convex optimization problem whenever there are inequality constraints in (1). Of course, one can still apply the semismooth Newton method to the optimality condition of problem (6) in a local sense. The problem is that the globalization techniques introduced in [34] are no longer valid (see section 2 for more discussions on this). On the other hand, the projected gradient method used by Boyd and Xiao [10] can converge at best linearly, while the rate of convergence of a BFGS type method suggested by Malick [29] is still an open question due to the fact that  $\theta(\cdot)$  fails to be twice continuously differentiable (for a discussion on the convergence of BFGS type methods for problems of such a nature, see Chen [12]). Thus, a natural question arises: can one still expect a fast convergent numerical method for solving the LSSDP (1) with both equality and inequality constraints? This paper will give an affirmative answer to this question by introducing a quadratically convergent inexact smoothing Newton method, whose ideas are largely borrowed from [37].

Just as the case for semismooth Newton methods, smoothing Newton methods for solving nonsmooth equations were mainly developed by the optimization community, in particular, the complementarity community (see [22] for an introduction on complementarity problems and on smoothing Newton methods), and have not been well received outside the optimization field. In this paper, we shall take the LSSDP problem (1) as an example to introduce smoothing Newton methods to the numerical linear algebra community.

The paper is organized as follows. In section 2, we present preliminaries on matrix-valued functions. Section 3 presents a general introduction on an inexact smoothing Newton method for solving nonsmooth equations and its convergence analysis. In section 4, we apply the introduced inexact smoothing Newton method to the LSSDP problem (1). We report our numerical results in section 5 and make our final conclusions in section 6.

**2. Preliminaries on matrix-valued functions.** For subsequent discussions, in this section we introduce some basic properties of matrix-valued functions related to the LSSDP problem (1) and its dual.

Let  $\mathcal{F}$  denote the feasible set of problem (1). Assume that  $\mathcal{F} \neq \emptyset$ . Then problem (1) has a unique optimal solution  $\bar{X}$ . Let  $q = m - p$  and  $\mathcal{Q} = \{0\}^p \times \mathbb{R}_+^q$ . Denote  $\mathcal{A} : \mathcal{S}^n \rightarrow \mathbb{R}^m$  by

$$\mathcal{A}(X) := \begin{bmatrix} \langle A_1, X \rangle \\ \vdots \\ \langle A_m, X \rangle \end{bmatrix}, \quad X \in \mathcal{S}^n.$$

For any symmetric  $X \in \mathcal{S}^n$ , we write  $X \succeq 0$  and  $X \succ 0$  to represent that  $X$  is positive semidefinite and positive definite, respectively. Then

$$\mathcal{F} = \{X \in \mathcal{S}^n \mid \mathcal{A}(X) \in b + \mathcal{Q}, X \succeq 0\},$$

and the dual problem of (1) takes the form

$$(8) \quad \begin{aligned} \min \quad & \theta(y) := \frac{1}{2} \|\Pi_{\mathcal{S}_+^n}(C + \mathcal{A}^*y)\|^2 - \langle b, y \rangle - \frac{1}{2} \|C\|^2 \\ \text{s.t.} \quad & y \in \mathcal{Q}^* = \mathbb{R}^p \times \mathbb{R}_+^q. \end{aligned}$$

The objective function  $\theta(\cdot)$  in (8) is a continuously differentiable convex function with

$$\nabla\theta(y) = \mathcal{A}\Pi_{\mathcal{S}_+^n}(C + \mathcal{A}^*y) - b, \quad y \in \mathbb{R}^m,$$

where the adjoint  $\mathcal{A}^* : \mathbb{R}^m \rightarrow \mathcal{S}^n$  takes the form

$$(9) \quad \mathcal{A}^*(y) = \sum_{i=1}^m y_i A_i, \quad y \in \mathbb{R}^m.$$

One classical dual approach described by Rockafellar in [42, page 4], when specialized to problem (1), is first to find an optimal solution  $\bar{y}$ , if it exists, to the dual problem (8) and then to obtain the unique optimal solution  $\bar{X}$  to problem (1) via  $\bar{X} = \Pi_{\mathcal{S}_+^n}(C + \mathcal{A}^*\bar{y})$ . See Malick [29] and Boyd and Xiao [10] for the worked out details.

In order to apply a dual based optimization method to solve problem (1), we need the following Slater condition to hold:

$$(10) \quad \begin{cases} \{A_i\}_{i=1}^p \text{ are linearly independent,} \\ \exists X^0 \in \mathcal{F} \text{ such that } \langle A_i, X^0 \rangle > b_i, \quad i = p + 1, \dots, m \text{ and } X^0 \succ 0. \end{cases}$$

The next proposition is a straightforward application of [42, Theorems 17 and 18].

PROPOSITION 2.1. *Under the Slater condition (10), the following hold:*

- (i) *There exists at least one  $\bar{y} \in \mathcal{Q}^*$  that solves the dual problem (8). The unique solution to problem (1) is given by*

$$(11) \quad \bar{X} = \Pi_{\mathcal{S}_+^n}(C + \mathcal{A}^*\bar{y}).$$

- (ii) *For every real number  $\tau$ , the constrained level set  $\{y \in \mathcal{Q}^* \mid \theta(y) \leq \tau\}$  is closed, bounded, and convex.*

Proposition 2.1 indicates that one should be able to use any gradient based optimization method to find an optimal solution to the convex problem (8), and thus to solve problem (1), as long as the Slater condition (10) holds. Note that for any given  $y \in \mathbb{R}^m$ , both  $\theta(y)$  and  $\nabla\theta(y)$  can be computed explicitly as the metric projector  $\Pi_{\mathcal{S}_+^n}(\cdot)$  has long been known by statisticians to admit an analytic formula [44].

Since  $\theta(\cdot)$  is a convex function,  $\bar{y} \in \mathcal{Q}^*$  solves problem (8) if and only if it satisfies the following variational inequality:

$$(12) \quad \langle y - \bar{y}, \nabla\theta(\bar{y}) \rangle \geq 0 \quad \forall y \in \mathcal{Q}^*.$$

Define  $F : \mathbb{R}^m \rightarrow \mathbb{R}^m$  by

$$(13) \quad F(y) := y - \Pi_{\mathcal{Q}^*}(y - \nabla\theta(y)), \quad y \in \mathbb{R}^m.$$

Then one can easily check that  $\bar{y} \in \mathcal{Q}^*$  solves (12) if and only if  $F(\bar{y}) = 0$  [21]. Thus, solving the dual problem (8) is equivalent to solving the following equation:

$$(14) \quad F(y) = 0, \quad y \in \mathbb{R}^m.$$

Since both  $\Pi_{\mathcal{Q}^*}$  and  $\Pi_{\mathcal{S}_+^n}$  are globally Lipschitz continuous,  $F$  is globally Lipschitz continuous. This means that though one cannot use the classical Newton method to solve (14), one can still use Clarke’s generalized Jacobian based Newton methods [27, 36, 38]. Unlike the case with equality constraints only, however,  $F(\cdot)$  is no longer the gradient mapping of any real valued function. This means that we cannot use the techniques in [34] to globalize Clarke’s generalized Jacobian based Newton methods. In this paper, we shall introduce an inexact smoothing Newton method to overcome this difficulty. For this purpose, we need smoothing functions for  $F(\cdot)$ .

Next, we shall first discuss smoothing functions for the metric projector  $\Pi_{\mathcal{S}_+^n}(\cdot)$ . Let  $X \in \mathcal{S}^n$ . Suppose that  $X$  has the spectral decomposition

$$(15) \quad X = P \text{diag}(\lambda_1, \dots, \lambda_n) P^T,$$

where  $\lambda_1 \geq \dots \geq \lambda_n$  are the eigenvalues of  $X$  and  $P$  is a corresponding orthogonal matrix of orthonormal eigenvectors of  $X$ . Then, from [44],

$$(16) \quad \Pi_{\mathcal{S}_+^n}(X) = P \text{diag}(\max(0, \lambda_1), \dots, \max(0, \lambda_n)) P^T.$$

Define

$$\alpha := \{i \mid \lambda_i > 0\}, \quad \beta := \{i \mid \lambda_i = 0\}, \quad \text{and} \quad \gamma := \{i \mid \lambda_i < 0\}.$$

Write  $P = [P_\alpha \ P_\beta \ P_\gamma]$  with  $P_\alpha$ ,  $P_\beta$ , and  $P_\gamma$  containing the columns in  $P$  indexed by  $\alpha$ ,  $\beta$ , and  $\gamma$ , respectively. Let  $\phi : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  be defined by the following Huber smoothing function:

$$(17) \quad \phi(\varepsilon, t) = \begin{cases} t & \text{if } t \geq \frac{\varepsilon}{2}, \\ \frac{1}{2\varepsilon} \left(t + \frac{\varepsilon}{2}\right)^2 & \text{if } -\frac{\varepsilon}{2} < t < \frac{\varepsilon}{2}, \\ 0 & \text{if } t \leq -\frac{\varepsilon}{2}. \end{cases} \quad (\varepsilon, t) \in \mathbb{R} \times \mathbb{R},$$

For any  $\varepsilon \in \mathbb{R}$ , let

$$(18) \quad \Phi(\varepsilon, X) := P \begin{bmatrix} \phi(\varepsilon, \lambda_1) & & \\ & \ddots & \\ & & \phi(\varepsilon, \lambda_n) \end{bmatrix} P^T.$$

Note that when  $\varepsilon = 0$ ,  $\Phi(0, X) = \Pi_{\mathcal{S}_+^n}(X)$ . By a famous result of Löwner [28], we know that when  $\varepsilon \neq 0$  or  $\beta = \emptyset$ ,

$$(19) \quad \Phi'_X(\varepsilon, X)(H) = P[\Omega(\varepsilon, \lambda) \circ (P^T H P)] P^T \quad \forall H \in \mathcal{S}^n,$$

where “ $\circ$ ” denotes the Hadamard product,  $\lambda = (\lambda_1, \dots, \lambda_n)^T$ , and the symmetric matrix  $\Omega(\varepsilon, \lambda)$  is given by

$$(20) \quad [\Omega(\varepsilon, \lambda)]_{ij} = \begin{cases} \frac{\phi(\varepsilon, \lambda_i) - \phi(\varepsilon, \lambda_j)}{\lambda_i - \lambda_j} \in [0, 1] & \text{if } \lambda_i \neq \lambda_j, \\ \phi'_{\lambda_i}(\varepsilon, \lambda_i) \in [0, 1] & \text{if } \lambda_i = \lambda_j, \end{cases} \quad i, j = 1, \dots, n.$$

When  $\varepsilon \neq 0$  or  $\beta = \emptyset$ , the partial derivative of  $\Phi(\cdot, \cdot)$  with respect to  $\varepsilon$  can be computed by

$$\Phi'_\varepsilon(\varepsilon, X) = P \text{diag}(\phi'_\varepsilon(\varepsilon, \lambda_1), \dots, \phi'_\varepsilon(\varepsilon, \lambda_n)) P^T.$$

Thus,  $\Phi(\cdot, \cdot)$  is continuously differentiable around  $(\varepsilon, X) \in \mathbb{R} \times \mathcal{S}^n$  if  $\varepsilon \neq 0$  or  $\beta = \emptyset$ . Furthermore,  $\Phi(\cdot, \cdot)$  is globally Lipschitz continuous and strongly semismooth at any  $(0, X) \in \mathbb{R} \times \mathcal{S}^n$  [52]. In particular, for any  $\varepsilon \downarrow 0$  and  $\mathcal{S}^n \ni H \rightarrow 0$ , it holds that

$$(21) \quad \Phi(\varepsilon, X + H) - \Phi(0, X) - \Phi'(\varepsilon, X + H)(\varepsilon, H) = O(\|(\varepsilon, H)\|^2).$$

Recall that for a locally Lipschitz continuous function  $\Gamma$  from a finite dimensional real Hilbert space  $\mathcal{X}$  to  $\mathbb{R}^n$ , the B-subdifferential of  $\Gamma$  at  $x \in \mathcal{X}$  in the sense of Qi [36] is defined by

$$\partial_B \Gamma(x) := \left\{ V \mid V = \lim_{k \rightarrow \infty} \Gamma'(x^k), x^k \rightarrow x, x_k \in \mathcal{D}_\Gamma \right\},$$

where  $\mathcal{D}_\Gamma$  is the set of points where  $\Gamma$  is Fréchet differentiable. The generalized Jacobian  $\partial \Gamma(x)$  of  $\Gamma$  at  $x$  in the sense of Clarke [13] is just the convex hull of  $\partial_B \Gamma(x)$ .

Define  $\Phi_{|\beta|} : \mathbb{R} \times \mathcal{S}^{|\beta|} \rightarrow \mathcal{S}^{|\beta|}$  by replacing the dimension  $n$  in the definition of  $\Phi : \mathbb{R} \times \mathcal{S}^n \rightarrow \mathcal{S}^n$  with  $|\beta|$ . As the case for  $\Phi(\cdot, \cdot)$ , the mapping  $\Phi_{|\beta|}(\cdot, \cdot)$  is also Lipschitz continuous. Then the B-subdifferentials  $\partial_B \Phi(0, X)$  of  $\Phi$  at  $(0, X)$  and  $\partial_B \Phi_{|\beta|}(0, Z)$  of  $\Phi_{|\beta|}$  at  $(0, Z) \in \mathbb{R} \times \mathcal{S}^{|\beta|}$  in the sense of Qi [36] are both well defined. The following result can be proven similarly as in [11, Proposition 5].

**PROPOSITION 2.2.** *Suppose that  $X \in \mathcal{S}^n$  has the spectral decomposition as in (15). Then  $V \in \partial_B \Phi(0, X)$  if and only if there exists  $V_{|\beta|} \in \partial_B \Phi_{|\beta|}(0, 0)$  such that for all  $(\varepsilon, H) \in \mathbb{R} \times \mathcal{S}^n$ ,*

$$(22) \quad V(\varepsilon, H) = P \begin{bmatrix} P_\alpha^T H P_\alpha & P_\alpha^T H P_\beta & U_{\alpha\gamma} \circ (P_\alpha^T H P_\gamma) \\ (P_\alpha^T H P_\beta)^T & V_{|\beta|}(\varepsilon, P_\beta^T H P_\beta) & 0 \\ (P_\alpha^T H P_\gamma)^T \circ U_{\alpha\gamma}^T & 0 & 0 \end{bmatrix} P^T,$$

where  $U \in \mathcal{S}^n$  is defined by

$$(23) \quad U_{ij} := \frac{\max\{\lambda_i, 0\} + \max\{\lambda_j, 0\}}{|\lambda_i| + |\lambda_j|}, \quad i, j = 1, \dots, n,$$

where  $0/0$  is defined to be 1.

In order to define smoothing functions for  $F(\cdot)$ , we need to define smoothing functions for  $\Pi_{\mathcal{Q}^*}(\cdot)$ . This, however, can be done in many different ways. For simplicity, we shall use only the function  $\phi$  given by (17) to define a smoothing function for  $\Pi_{\mathcal{Q}^*}(\cdot)$ . Let  $\psi : \mathbb{R} \times \mathbb{R}^m \rightarrow \mathbb{R}^m$  be defined by

$$(24) \quad \psi_i(\varepsilon, z) = \begin{cases} z_i & \text{if } i = 1, \dots, p, \\ \phi(\varepsilon, z_i) & \text{if } i = p + 1, \dots, m, \end{cases} \quad (\varepsilon, z) \in \mathbb{R} \times \mathbb{R}^m.$$

The function  $\psi$  is obviously continuously differentiable around any  $(\varepsilon, z) \in \mathbb{R} \times \mathbb{R}^m$  as long as  $\varepsilon \neq 0$  and is strongly semismooth everywhere.

Now, we are ready to define a smoothing function for  $F(\cdot)$  itself. Let

$$(25) \quad \Upsilon(\varepsilon, y) := y - \psi(\varepsilon, y - (\mathcal{A}\Phi(\varepsilon, C + \mathcal{A}^*y) - b)), \quad (\varepsilon, y) \in \mathbb{R} \times \mathbb{R}^m.$$

By the definitions of  $\Upsilon$ ,  $\psi$ , and  $\Phi$ , we know that for any  $y \in \mathbb{R}^m$ ,  $F(y) = \Upsilon(0, y)$ . We summarize several useful properties of  $\Upsilon$  in the next proposition.

**PROPOSITION 2.3.** *Let  $\Upsilon : \mathbb{R} \times \mathbb{R}^m$  be defined by (25). Let  $y \in \mathbb{R}^m$ . Then the following hold:*

- (i)  $\Upsilon$  is globally Lipschitz continuous on  $\mathbb{R} \times \mathbb{R}^m$ .
- (ii)  $\Upsilon$  is continuously differentiable around  $(\varepsilon, y)$  when  $\varepsilon \neq 0$ . For any fixed  $\varepsilon \in \mathbb{R}$ ,  $\Upsilon(\varepsilon, \cdot)$  is a  $P_0$ -function; i.e., for any  $(y, h) \in \mathbb{R}^m \times \mathbb{R}^m$  with  $y \neq h$ ,

$$(26) \quad \max_{y_i \neq h_i} (y_i - h_i)(\Upsilon_i(\varepsilon, y) - \Upsilon_i(\varepsilon, h)) \geq 0,$$

and thus for any fixed  $\varepsilon \neq 0$ ,  $\Upsilon'_y(\varepsilon, y)$  is a  $P_0$ -matrix.

- (iii)  $\Upsilon$  is strongly semismooth at  $(0, y)$ . In particular, for any  $\varepsilon \downarrow 0$  and  $\mathbb{R}^m \ni h \rightarrow 0$  we have

$$\Upsilon(\varepsilon, y + h) - \Upsilon(0, y) - \Upsilon'(\varepsilon, y + h) \begin{pmatrix} \varepsilon \\ h \end{pmatrix} = O(\|(\varepsilon, h)\|^2).$$

- (iv) For any  $h \in \mathbb{R}^m$ ,

$$\partial_B \Upsilon(0, y)(0, h) \subseteq h - \partial_B \psi(0, y - \nabla \theta(y))(0, h - \mathcal{A} \partial_B \Phi(0, C + \mathcal{A}^* y)(0, \mathcal{A}^* h)).$$

*Proof.* (i) Since both  $\psi$  and  $\Phi$  are globally Lipschitz continuous,  $\Upsilon$  is also globally Lipschitz continuous.

(ii) From the definitions of  $\psi$  and  $\Phi$  we know that  $\Upsilon$  is continuously differentiable around  $(\varepsilon, y) \in \mathbb{R} \times \mathbb{R}^m$  when  $\varepsilon \neq 0$ .

Since, by part (i),  $\Upsilon$  is continuous on  $\mathbb{R} \times \mathbb{R}^m$ , we need only to show that for any  $0 \neq \varepsilon \in \mathbb{R}$ ,  $\Upsilon(\varepsilon, \cdot)$  is a  $P_0$ -function.

Fix  $\varepsilon \neq 0$ . Define  $g_\varepsilon : \mathbb{R}^m \rightarrow \mathbb{R}^m$  by

$$g_\varepsilon(y) = \mathcal{A} \Phi(\varepsilon, C + \mathcal{A}^* y) - b, \quad y \in \mathbb{R}^m.$$

Then  $g_\varepsilon$  is continuously differentiable on  $\mathbb{R}^m$ . From (19) and (20), we have

$$\langle h, (g_\varepsilon)'(y)h \rangle = \langle h, \mathcal{A} \Phi'_X(\varepsilon, X)(\mathcal{A}^* h) \rangle = \langle \mathcal{A}^* h, \Phi'_X(\varepsilon, X)(\mathcal{A}^* h) \rangle \geq 0 \quad \forall h \in \mathbb{R}^m,$$

which implies that  $g_\varepsilon$  is a  $P_0$ -function on  $\mathbb{R}^m$ . Let  $(y, h) \in \mathbb{R}^m \times \mathbb{R}^m$  with  $y \neq h$ . Then there exists  $i \in \{1, \dots, m\}$  with  $y_i \neq h_i$  such that

$$(y_i - h_i)((g_\varepsilon)_i(y) - (g_\varepsilon)_i(h)) \geq 0.$$

Furthermore, by noting that for any  $z \in \mathbb{R}^m$ ,

$$\phi'_{z_i}(\varepsilon, z_i) \in [0, 1], \quad i = 1, \dots, m,$$

we obtain that

$$(y_i - h_i)(\Upsilon_i(\varepsilon, y) - \Upsilon_i(\varepsilon, h)) \geq 0.$$

This shows that (26) holds. Thus,  $\Upsilon'_y(\varepsilon, y)$  is a  $P_0$ -matrix for any fixed  $\varepsilon \neq 0$ .

(iii) Since it can be checked directly that the composite of strongly semismooth functions is still strongly semismooth [23],  $\Upsilon$  is strongly semismooth at  $(0, y)$ .

(iv) Since both  $\psi$  and  $\Phi$  are directionally differentiable, for any  $(\varepsilon, y') \in \mathbb{R} \times \mathbb{R}^m$  such that  $\Upsilon$  is Fréchet differentiable at  $(\varepsilon, y')$ ,

$$\Upsilon'(\varepsilon, y')(0, h) = h - \psi'((\varepsilon, z'); (0, h - \mathcal{A} \Phi'((\varepsilon, C + \mathcal{A}^* y'); (0, \mathcal{A}^* h)))) ,$$

which, together with the semismoothness of  $\psi$  and  $\Phi$ , implies

$$\Upsilon'(\varepsilon, y')(0, h) \in h - \partial_B \psi(\varepsilon, z')(0, h - \mathcal{A} \partial_B \Phi(\varepsilon, C + \mathcal{A}^* y')(0, \mathcal{A}^* h)),$$

where  $z' := y' - (\mathcal{A} \Phi(\varepsilon, C + \mathcal{A}^* y') - b)$ . By taking  $(\varepsilon, y') \rightarrow (0, y)$  in the above inclusion, we complete the proof.  $\square$



**3. An inexact smoothing Newton method.** The purpose of this section is to introduce an inexact smoothing Newton method for solving the general nonsmooth equation

$$F(y) = 0, \quad y \in \mathbb{R}^m,$$

where  $F : \mathbb{R}^m \rightarrow \mathbb{R}^m$  is a locally Lipschitz continuous function, which is not necessarily the mapping defined in the last section. This inexact smoothing Newton method is largely modified from the exact smoothing Newton method constructed in [37] for solving complementarity and variational inequality problems. The motivation to introduce an inexact version is completely from the computational point of view because the costs of the exact smoothing Newton method for solving problems such as the LSSDP problem (1) are prohibitive. We shall talk more about this in the numerical section.

Let  $G : \mathbb{R} \times \mathbb{R}^m \rightarrow \mathbb{R}^m$  be a locally Lipschitz continuous function satisfying

$$G(\varepsilon, y') \rightarrow F(y) \quad \text{as} \quad (\varepsilon, y') \rightarrow (0, y).$$

Furthermore,  $G$  is required to be continuously differentiable around any  $(\varepsilon, y)$  unless  $\varepsilon = 0$ . The existence of such a function  $G$  can be easily proven via convolution. Define  $E : \mathbb{R} \times \mathbb{R}^m \rightarrow \mathbb{R} \times \mathbb{R}^m$  by

$$E(\varepsilon, y) := \begin{bmatrix} \varepsilon \\ G(\varepsilon, y) \end{bmatrix}, \quad (\varepsilon, y) \in \mathbb{R} \times \mathbb{R}^m.$$

Then solving the nonsmooth equation  $F(y) = 0$  is equivalent to solving the following smoothing-nonsmooth equation:

$$E(\varepsilon, y) = 0.$$

Our inexact smoothing Newton method is specifically designed for solving the latter.

Define the merit function  $\varphi : \mathbb{R} \times \mathbb{R}^m \rightarrow \mathbb{R}_+$  by

$$\varphi(\varepsilon, y) := \|E(\varepsilon, y)\|^2, \quad (\varepsilon, y) \in \mathbb{R} \times \mathbb{R}^m.$$

Choose  $r \in (0, 1)$ . Let

$$\zeta(\varepsilon, y) := r \min\{1, \varphi(\varepsilon, y)\}, \quad (\varepsilon, y) \in \mathbb{R} \times \mathbb{R}^m.$$

Then the inexact smoothing Newton method can be described as follows.

**ALGORITHM 3.1** (an inexact smoothing Newton method).

**Step 0.** Let  $\hat{\varepsilon} \in (0, \infty)$  and  $\eta \in (0, 1)$  be such that

$$\delta := \sqrt{2} \max\{r\hat{\varepsilon}, \eta\} < 1.$$

Select constants  $\rho \in (0, 1)$ ,  $\sigma \in (0, 1/2)$ ,  $\tau \in (0, 1)$ , and  $\hat{\tau} \in [1, \infty)$ . Let  $\varepsilon^0 := \hat{\varepsilon}$  and  $y^0 \in \mathbb{R}^m$  be an arbitrary point.  $k := 0$ .

**Step 1.** If  $E(\varepsilon^k, y^k) = 0$ , then stop. Otherwise, compute

$$\zeta_k := r \min\{1, \varphi(\varepsilon^k, y^k)\} \quad \text{and} \quad \eta_k := \min\{\tau, \hat{\tau} \|E(\varepsilon^k, y^k)\|\}.$$

**Step 2.** Solve the equation

$$(27) \quad E(\varepsilon^k, y^k) + E'(\varepsilon^k, y^k) \begin{bmatrix} \Delta \varepsilon^k \\ \Delta y^k \end{bmatrix} = \begin{bmatrix} \zeta_k \hat{\varepsilon} \\ 0 \end{bmatrix}$$

approximately such that

$$(28) \quad \|R_k\| \leq \min\{\eta_k \|G(\varepsilon^k, y^k) + G'_\varepsilon(\varepsilon^k, y^k)\Delta\varepsilon^k\|, \eta \|E(\varepsilon^k, y^k)\|\},$$

where

$$\Delta\varepsilon^k := -\varepsilon^k + \zeta_k \hat{\varepsilon}$$

and

$$R_k := G(\varepsilon^k, y^k) + G'(\varepsilon^k, y^k) \begin{bmatrix} \Delta\varepsilon^k \\ \Delta y^k \end{bmatrix}.$$

**Step 3.** Let  $l_k$  be the smallest nonnegative integer  $l$  satisfying

$$(29) \quad \varphi(\varepsilon^k + \rho^l \Delta\varepsilon^k, y^k + \rho^l \Delta y^k) \leq [1 - 2\sigma(1 - \delta)\rho^l] \varphi(\varepsilon^k, y^k).$$

Define

$$(\varepsilon^{k+1}, y^{k+1}) := (\varepsilon^k + \rho^{l_k} \Delta\varepsilon^k, y^k + \rho^{l_k} \Delta y^k).$$

**Step 4.** Replace  $k$  by  $k + 1$  and go to **Step 1**.

LEMMA 3.2. If for some  $(\tilde{\varepsilon}, \tilde{y}) \in \mathbb{R}_{++} \times \mathbb{R}^m$ ,  $E'(\tilde{\varepsilon}, \tilde{y})$  is nonsingular, then there exist an open neighborhood  $\mathcal{O}$  of  $(\tilde{\varepsilon}, \tilde{y})$  and a positive number  $\bar{\alpha} \in (0, 1]$  such that for any  $(\varepsilon, y) \in \mathcal{O}$  and  $\alpha \in [0, \bar{\alpha}]$ ,  $\varepsilon \in \mathbb{R}_{++}$ ,  $E'(\varepsilon, y)$  is nonsingular, and

$$(30) \quad \varphi(\varepsilon + \alpha\Delta\varepsilon, y + \alpha\Delta y) \leq [1 - 2\sigma(1 - \delta)\alpha] \varphi(\varepsilon, y),$$

where  $(\Delta\varepsilon, \Delta y) \in \mathbb{R} \times \mathbb{R}^m$  satisfies

$$\Delta\varepsilon = -\varepsilon + \zeta(\varepsilon, y)\hat{\varepsilon}$$

and

$$\left\| G(\varepsilon, y) + G'(\varepsilon, y) \begin{bmatrix} \Delta\varepsilon \\ \Delta y \end{bmatrix} \right\| \leq \eta \|E(\varepsilon, y)\|.$$

*Proof.* Since  $\tilde{\varepsilon} \in \mathbb{R}_{++}$  and  $E'(\tilde{\varepsilon}, \tilde{y})$  is nonsingular, there exists an open neighborhood  $\mathcal{O}$  of  $(\tilde{\varepsilon}, \tilde{y})$  such that for any  $(\varepsilon, y) \in \mathcal{O}$ ,  $\varepsilon \in \mathbb{R}_{++}$  and  $E'(\varepsilon, y)$  is nonsingular.

For any  $(\varepsilon, y) \in \mathcal{O}$ , denote

$$R(\varepsilon, y) := G(\varepsilon, y) + G'(\varepsilon, y) \begin{bmatrix} \Delta\varepsilon \\ \Delta y \end{bmatrix}.$$

Then  $(\Delta\varepsilon, \Delta y)$  is the unique solution of the following equation:

$$E(\varepsilon, y) + E'(\varepsilon, y) \begin{bmatrix} \Delta\varepsilon \\ \Delta y \end{bmatrix} = \begin{bmatrix} \zeta(\varepsilon, y)\hat{\varepsilon} \\ R(\varepsilon, y) \end{bmatrix}.$$

Thus,

$$\begin{aligned} & \left\langle \nabla\varphi(\varepsilon, y), \begin{bmatrix} \Delta\varepsilon \\ \Delta y \end{bmatrix} \right\rangle = \left\langle 2\nabla E(\varepsilon, y)E(\varepsilon, y), \begin{bmatrix} \Delta\varepsilon \\ \Delta y \end{bmatrix} \right\rangle \\ & = \left\langle 2E(\varepsilon, y), \begin{bmatrix} \zeta(\varepsilon, y)\hat{\varepsilon} \\ R(\varepsilon, y) \end{bmatrix} - E(\varepsilon, y) \right\rangle \\ & = -2\varphi(\varepsilon, y) + 2\varepsilon\zeta(\varepsilon, y)\hat{\varepsilon} + 2\langle R(\varepsilon, y), G(\varepsilon, y) \rangle \\ & \leq -2\varphi(\varepsilon, y) + 2\varepsilon(r\hat{\varepsilon}) \min\{1, \varphi(\varepsilon, y)\} + 2\eta\varphi(\varepsilon, y)^{1/2}\|G(\varepsilon, y)\|, \end{aligned}$$

which implies that if  $\varphi(\varepsilon, y) > 1$ , we have

$$\begin{aligned}
 & \left\langle \nabla\varphi(\varepsilon, y), \begin{bmatrix} \Delta\varepsilon \\ \Delta y \end{bmatrix} \right\rangle \\
 & \leq -2\varphi(\varepsilon, y) + 2\varepsilon(r\hat{\varepsilon}) + 2\eta\varphi(\varepsilon, y)^{1/2}\|G(\varepsilon, y)\| \\
 & \leq -2\varphi(\varepsilon, y) + 2\max\{r\hat{\varepsilon}, \eta\}(\varepsilon + \varphi(\varepsilon, y)^{1/2}\sqrt{\varphi(\varepsilon, y) - \varepsilon^2}) \\
 & \leq -2\varphi(\varepsilon, y) + 2\sqrt{2}\max\{r\hat{\varepsilon}, \eta\}\varphi(\varepsilon, y) \\
 (31) \quad & = 2(\sqrt{2}\max\{r\hat{\varepsilon}, \eta\} - 1)\varphi(\varepsilon, y)
 \end{aligned}$$

and if  $\varphi(\varepsilon, y) < 1$ , we have

$$\begin{aligned}
 & \left\langle \nabla\varphi(\varepsilon, y), \begin{bmatrix} \Delta\varepsilon \\ \Delta y \end{bmatrix} \right\rangle \\
 & \leq -2\varphi(\varepsilon, y) + 2\varepsilon(r\hat{\varepsilon})\varphi(\varepsilon, y) + 2\eta\varphi(\varepsilon, y)^{1/2}\|G(\varepsilon, y)\| \\
 & \leq -2\varphi(\varepsilon, y) + 2\max\{r\hat{\varepsilon}, \eta\}\varphi(\varepsilon, y)^{1/2}(\varepsilon\varphi(\varepsilon, y)^{1/2} + \sqrt{\varphi(\varepsilon, y) - \varepsilon^2}) \\
 & \leq -2\varphi(\varepsilon, y) + 2\sqrt{2}\max\{r\hat{\varepsilon}, \eta\}\varphi(\varepsilon, y) \\
 (32) \quad & = 2(\sqrt{2}\max\{r\hat{\varepsilon}, \eta\} - 1)\varphi(\varepsilon, y).
 \end{aligned}$$

Therefore, by inequalities (31) and (32), we have

$$(33) \quad \left\langle \nabla\varphi(\varepsilon, y), \begin{bmatrix} \Delta\varepsilon \\ \Delta y \end{bmatrix} \right\rangle \leq -2(1 - \delta)\varphi(\varepsilon, y).$$

By using the fact that  $\nabla\varphi(\cdot, \cdot)$  is uniformly continuous on  $\mathcal{O}$ , we obtain from the Taylor expansion that

$$\varphi(\varepsilon + \alpha\Delta\varepsilon, y + \alpha\Delta y) = \varphi(\varepsilon, y) + \alpha \left\langle \nabla\varphi(\varepsilon, y), \begin{bmatrix} \Delta\varepsilon \\ \Delta y \end{bmatrix} \right\rangle + o(\alpha) \quad \forall (\varepsilon, y) \in \mathcal{O},$$

which, together with (33), implies that there exists a positive number  $\bar{\alpha} \in (0, 1]$  such that for all  $\alpha \in [0, \bar{\alpha}]$ , (30) holds.  $\square$

Let

$$(34) \quad \mathcal{N} := \{(\varepsilon, y) \mid \varepsilon \geq \zeta(\varepsilon, y)\hat{\varepsilon}\}.$$

**PROPOSITION 3.3.** *For each fixed  $k \geq 0$ , if  $\varepsilon^k \in \mathbb{R}_{++}$ ,  $(\varepsilon^k, y^k) \in \mathcal{N}$ , and  $E'(\varepsilon^k, y^k)$  is nonsingular, then for any  $\alpha \in [0, 1]$  such that*

$$(35) \quad \varphi(\varepsilon^k + \alpha\Delta\varepsilon^k, y^k + \alpha\Delta y^k) \leq [1 - 2\sigma(1 - \delta)\alpha]\varphi(\varepsilon^k, y^k)$$

it holds that  $(\varepsilon^k + \alpha\Delta\varepsilon^k, y^k + \alpha\Delta y^k) \in \mathcal{N}$ .

*Proof.* Note that  $(\varepsilon^k, y^k) \in \mathcal{N}$ , i.e.,  $\varepsilon^k \geq \zeta_k\hat{\varepsilon}$ , so  $\Delta\varepsilon^k = -\varepsilon^k + \zeta_k\hat{\varepsilon} \leq 0$ . Thus, by the definition of  $\zeta$ , together with (35), we have

$$\begin{aligned}
 & \varepsilon^k + \alpha\Delta\varepsilon^k - \zeta(\varepsilon^k + \alpha\Delta\varepsilon^k, y^k + \alpha\Delta y^k)\hat{\varepsilon} \\
 & \geq \varepsilon^k + \Delta\varepsilon^k - \zeta(\varepsilon^k + \alpha\Delta\varepsilon^k, y^k + \alpha\Delta y^k)\hat{\varepsilon} \\
 & = \zeta_k\hat{\varepsilon} - \zeta(\varepsilon^k + \alpha\Delta\varepsilon^k, y^k + \alpha\Delta y^k)\hat{\varepsilon} \\
 (36) \quad & \geq 0.
 \end{aligned}$$

This completes our proof.  $\square$

In order to discuss the global convergence of Algorithm 3.1 we need the following assumption.

*Assumption 3.4.* For any  $(\varepsilon, y) \in \mathbb{R}_{++} \times \mathbb{R}^n$ ,  $E'(\varepsilon, y)$  is nonsingular.

**THEOREM 3.5.** *Suppose that Assumption 3.4 is satisfied. Then Algorithm 3.1 is well defined and generates an infinite sequence  $\{(\varepsilon^k, y^k)\} \in \mathcal{N}$  with the property that any accumulation point  $(\bar{\varepsilon}, \bar{y})$  of  $\{(\varepsilon^k, y^k)\}$  is a solution of  $E(\varepsilon, y) = 0$ .*

*Proof.* It follows from Lemma 3.2, Proposition 3.3, and Assumption 3.4 that Algorithm 3.1 is well defined and generates an infinite sequence  $\{(\varepsilon^k, y^k)\} \in \mathcal{N}$ .

From the design of Algorithm 3.1,  $\varphi(\varepsilon^{k+1}, y^{k+1}) < \varphi(\varepsilon^k, y^k)$  for all  $k \geq 0$ . Hence, the two sequences  $\{\varphi(\varepsilon^k, y^k)\}$  and  $\{\zeta(\varepsilon^k, y^k)\}$  are monotonically decreasing. Since both  $\varphi(\varepsilon^k, y^k)$  and  $\zeta(\varepsilon^k, y^k)$  are nonnegative for  $k \geq 0$ , there exist  $\bar{\varphi} \geq 0$  and  $\bar{\zeta} \geq 0$  such that  $\varphi(\varepsilon^k, y^k) \rightarrow \bar{\varphi}$  and  $\zeta(\varepsilon^k, y^k) \rightarrow \bar{\zeta}$  as  $k \rightarrow \infty$ .

Let  $(\bar{\varepsilon}, \bar{y})$  be any accumulation point (if it exists) of  $\{(\varepsilon^k, y^k)\}$ . By taking a subsequence if necessary, we may assume that  $\{(\varepsilon^k, y^k)\}$  converges to  $(\bar{\varepsilon}, \bar{y})$ . Then  $\bar{\varphi} = \varphi(\bar{\varepsilon}, \bar{y})$ ,  $\bar{\zeta} = \zeta(\bar{\varepsilon}, \bar{y})$ , and  $(\bar{\varepsilon}, \bar{y}) \in \mathcal{N}$ .

Suppose that  $\bar{\varphi} > 0$ . Then, from  $\zeta(\bar{\varepsilon}, \bar{y}) = r \min\{1, \varphi(\bar{\varepsilon}, \bar{y})\}$  and  $(\bar{\varepsilon}, \bar{y}) \in \mathcal{N}$ , we see that  $\bar{\varepsilon} \in \mathbb{R}_{++}$ . Thus, from Assumption 3.4,  $E'(\bar{\varepsilon}, \bar{y})$  exists and is invertible. Hence, from Lemma 3.2, there exist an open neighborhood  $\mathcal{O}$  of  $(\bar{\varepsilon}, \bar{y})$  and a positive number  $\bar{\alpha} \in (0, 1]$  such that for any  $(\varepsilon, y) \in \mathcal{O}$  and all  $\alpha \in [0, \bar{\alpha}]$ ,  $\varepsilon \in \mathbb{R}_{++}$ ,  $E'(\varepsilon, y)$  is invertible, and (30) holds. Therefore, there exists a nonnegative integer  $l$  such that  $\rho^l \in (0, \bar{\alpha}]$  and  $\rho^{lk} \geq \rho^l$  for all  $k$  sufficiently large. Thus,

$$\varphi(\varepsilon^{k+1}, y^{k+1}) \leq [1 - 2\sigma(1 - \delta)\rho^{lk}]\varphi(\varepsilon^k, y^k) \leq [1 - 2\sigma(1 - \delta)\rho^l]\varphi(\varepsilon^k, y^k)$$

for all sufficiently large  $k$ . This contradicts the fact that the sequence  $\{\varphi(\varepsilon^k, y^k)\}$  converges to  $\bar{\varphi} > 0$ . This contradiction shows that  $\varphi(\bar{\varepsilon}, \bar{y}) = \bar{\varphi} = 0$ . That is,  $E(\bar{\varepsilon}, \bar{y}) = 0$ . The proof is completed.  $\square$

**THEOREM 3.6.** *Suppose that Assumption 3.4 is satisfied and that  $(\bar{\varepsilon}, \bar{y})$  is an accumulation point of the infinite sequence  $\{(\varepsilon^k, y^k)\}$  generated by Algorithm 3.1. Suppose that  $E$  is strongly semismooth at  $(\bar{\varepsilon}, \bar{y})$  and that all  $V \in \partial_B E(\bar{\varepsilon}, \bar{y})$  are nonsingular. Then the whole sequence  $\{(\varepsilon^k, y^k)\}$  converges to  $(\bar{\varepsilon}, \bar{y})$  quadratically, i.e.,*

$$(37) \quad \|(\varepsilon^{k+1} - \bar{\varepsilon}, y^{k+1} - \bar{y})\| = O(\|(\varepsilon^k - \bar{\varepsilon}, y^k - \bar{y})\|^2).$$

*Proof.* First, from Theorem 3.5,  $(\bar{\varepsilon}, \bar{y})$  is a solution of  $E(\varepsilon, y) = 0$ . Then, since all  $V \in \partial_B E(\bar{\varepsilon}, \bar{y})$  are nonsingular, from [36], for all  $(\varepsilon^k, y^k)$  sufficiently close to  $(\bar{\varepsilon}, \bar{y})$ ,

$$\|E'(\varepsilon^k, y^k)^{-1}\| = O(1)$$

and

$$(38) \quad \begin{aligned} & \left\| \begin{pmatrix} \varepsilon^k \\ y^k \end{pmatrix} + \begin{pmatrix} \Delta\varepsilon^k \\ \Delta y^k \end{pmatrix} - \begin{pmatrix} \bar{\varepsilon} \\ \bar{y} \end{pmatrix} \right\| \\ &= \left\| \begin{pmatrix} \varepsilon^k \\ y^k \end{pmatrix} + E'(\varepsilon^k, y^k)^{-1} \left[ \begin{pmatrix} r\varphi(\varepsilon^k, y^k)\hat{\varepsilon} \\ R_k \end{pmatrix} - E(\varepsilon^k, y^k) \right] - \begin{pmatrix} \bar{\varepsilon} \\ \bar{y} \end{pmatrix} \right\| \\ &= \left\| -E'(\varepsilon^k, y^k)^{-1} \left[ E(\varepsilon^k, y^k) - E'(\varepsilon^k, y^k) \begin{pmatrix} \varepsilon^k - \bar{\varepsilon} \\ y^k - \bar{y} \end{pmatrix} - \begin{pmatrix} r\varphi(\varepsilon^k, y^k)\hat{\varepsilon} \\ R_k \end{pmatrix} \right] \right\| \\ &= O \left( \left\| E(\varepsilon^k, y^k) - E(\bar{\varepsilon}, \bar{y}) - E'(\varepsilon^k, y^k) \begin{pmatrix} \varepsilon^k - \bar{\varepsilon} \\ y^k - \bar{y} \end{pmatrix} \right\| \right) \\ & \quad + O(\varphi(\varepsilon^k, y^k)) + O(\|R_k\|). \end{aligned}$$

Since  $E$  is locally Lipschitz continuous near  $(\bar{\varepsilon}, \bar{y})$ , for all  $(\varepsilon^k, y^k)$  close to  $(\bar{\varepsilon}, \bar{y})$  we have

$$(39) \quad \varphi(\varepsilon^k, y^k) = \|E(\varepsilon^k, y^k) - E(\bar{\varepsilon}, \bar{y})\|^2 = O(\|(\varepsilon^k - \bar{\varepsilon}, y^k - \bar{y})\|^2)$$

and

$$(40) \quad \begin{aligned} \|R_k\| &\leq \eta_k \|G(\varepsilon^k, y^k) + G'_\varepsilon(\varepsilon^k, y^k) \Delta \varepsilon^k\| \\ &\leq O(\|E(\varepsilon^k, y^k)\|) (\|G(\varepsilon^k, y^k)\| + O(|\Delta \varepsilon^k|)) \\ &\leq O(\|E(\varepsilon^k, y^k) - E(\bar{\varepsilon}, \bar{y})\|^2). \end{aligned}$$

Therefore, by using the assumption that  $E$  is strongly semismooth at  $(\bar{\varepsilon}, \bar{y})$  and the relations (38), (39), and (40), we have for all  $(\varepsilon^k, y^k)$  sufficiently close to  $(\bar{\varepsilon}, \bar{y})$  that

$$(41) \quad \|(\varepsilon^k, y^k) + (\Delta \varepsilon^k, \Delta y^k) - (\bar{\varepsilon}, \bar{y})\| = O(\|(\varepsilon^k, y^k) - (\bar{\varepsilon}, \bar{y})\|^2).$$

Finally, since  $E$  is strongly semismooth at  $(\bar{\varepsilon}, \bar{y})$  and all  $V \in \partial_B E(\bar{\varepsilon}, \bar{y})$  are nonsingular, we have for all  $(\varepsilon^k, y^k)$  sufficiently close to  $(\bar{\varepsilon}, \bar{y})$  that

$$\|(\varepsilon^k, y^k) - (\bar{\varepsilon}, \bar{y})\| \leq O(\|E(\varepsilon^k, y^k)\|),$$

which, together with (41) and the Lipschitz continuity of  $E$ , implies that

$$\varphi(\varepsilon^k + \Delta \varepsilon^k, y^k + \Delta y^k) = O(\varphi^2(\varepsilon^k, y^k)).$$

This shows that for all  $(\varepsilon^k, y^k)$  sufficiently close to  $(\bar{\varepsilon}, \bar{y})$ ,

$$(\varepsilon^{k+1}, y^{k+1}) = (\varepsilon^k, y^k) + (\Delta \varepsilon^k, \Delta y^k).$$

Thus, by using (41) we know that (37) holds.  $\square$

**4. The LSSDP.** In this section, we apply the general inexact smoothing Newton method developed in the last section to the LSSDP (1).

Let  $F : \mathbb{R}^m \rightarrow \mathbb{R}^m$  be defined by (13). Let  $\kappa \in (0, \infty)$  be a constant. Define  $G : \mathbb{R} \times \mathbb{R}^m \rightarrow \mathbb{R}^m$  by

$$(42) \quad G(\varepsilon, y) := \Upsilon(\varepsilon, y) + \kappa|\varepsilon|y, \quad (\varepsilon, y) \in \mathbb{R} \times \mathbb{R}^m,$$

where  $\Upsilon : \mathbb{R} \times \mathbb{R}^m \rightarrow \mathbb{R}^m$  is defined by (25). The reason for defining  $G$  by (42) is that for any  $(\varepsilon, y) \in \mathbb{R} \times \mathbb{R}^m$  with  $\varepsilon \neq 0$ ,  $G'_y(\varepsilon, y)$  is a  $P$ -matrix (i.e., all its principal minors are positive), thus nonsingular, while by part (ii) of Proposition 2.3,  $\Upsilon'_y(\varepsilon, y)$  is only a  $P_0$ -matrix (i.e., all its principal minors are nonnegative), which may be singular.

Let  $E : \mathbb{R} \times \mathbb{R}^m \rightarrow \mathbb{R} \times \mathbb{R}^m$  be defined by

$$(43) \quad E(\varepsilon, y) := \begin{bmatrix} \varepsilon \\ G(\varepsilon, y) \end{bmatrix} = \begin{bmatrix} \varepsilon \\ \Upsilon(\varepsilon, y) + \kappa|\varepsilon|y \end{bmatrix}, \quad (\varepsilon, y) \in \mathbb{R} \times \mathbb{R}^m.$$

Let  $\mathcal{N}$  be defined by (34). Next, we discuss convergent properties of Algorithm 3.1 when it is applied to solve  $E(\varepsilon, y) = 0$ .

**THEOREM 4.1.** *Algorithm 3.1 is well defined and generates an infinite sequence  $\{(\varepsilon^k, y^k)\} \in \mathcal{N}$  with the properties that any accumulation point  $(\bar{\varepsilon}, \bar{y})$  of  $\{(\varepsilon^k, y^k)\}$  is a solution of  $E(\varepsilon, y) = 0$  and  $\lim_{k \rightarrow \infty} \varphi(\varepsilon^k, y^k) = 0$ . Additionally, if the Slater condition (10) holds, then  $\{(\varepsilon^k, y^k)\}$  is bounded.*

*Proof.* From part (ii) of Proposition 2.3 and the definitions of  $G$  and  $E$  we know that for any  $(\varepsilon, y) \in \mathbb{R}_{++} \times \mathbb{R}^m$ ,  $G'_y(\varepsilon, y)$ , and so  $E'(\varepsilon, y)$  is a  $P$ -matrix. Then from Theorem 3.5 we know that Algorithm 3.1 is well defined and generates an infinite sequence  $\{(\varepsilon^k, y^k)\} \in \mathcal{N}$  with the property that any accumulation point  $(\bar{\varepsilon}, \bar{y})$  of  $\{(\varepsilon^k, y^k)\}$  is a solution of  $E(\varepsilon, y) = 0$ .

Since  $\varphi(\varepsilon^k, y^k)$  is a decreasing sequence,  $\lim_{k \rightarrow \infty} \varphi(\varepsilon^k, y^k)$  exists. Let

$$\bar{\varphi} := \lim_{k \rightarrow \infty} \varphi(\varepsilon^k, y^k) \geq 0.$$

If  $\bar{\varphi} > 0$ , then there exists an  $\varepsilon' > 0$  such that  $\varepsilon^k \geq \varepsilon'$  for all  $k \geq 0$ . For any  $v \geq 0$ , let

$$L_v := \{y \in \mathbb{R}^m \mid \|\Upsilon(\nu, y) + \kappa\nu y\| \leq v, \nu \in [\varepsilon', \hat{\varepsilon}]\}.$$

Then it is not difficult to prove that for any  $v \geq 0$ ,  $L_v$  is bounded. In fact, suppose that for some  $v \geq 0$ ,  $L_v$  is unbounded. Then there exist two sequences  $\{z^l\}$  and  $\{\nu^l\}$  such that  $\lim_{l \rightarrow \infty} \|z^l\| = \infty$  and for all  $l \geq 1$ ,  $\varepsilon' \leq \nu^l \leq \hat{\varepsilon}$  and  $\|\Upsilon(\nu^l, z^l) + \kappa\nu^l z^l\| \leq v$ . By taking subsequences if necessary, we may assume that  $\lim_{l \rightarrow \infty} \nu^l = \bar{\nu} \in [\varepsilon', \hat{\varepsilon}]$  and

$$i \in I^\infty \cup I^{-\infty} \cup I^v \quad \forall i \in \{1, \dots, m\},$$

where

$$\begin{aligned} I^\infty &:= \left\{ i \mid \lim_{l \rightarrow \infty} z_i^l = \infty, i = 1, \dots, m \right\}, \\ I^{-\infty} &:= \left\{ i \mid \lim_{l \rightarrow \infty} z_i^l = -\infty, i = 1, \dots, m \right\}, \quad \text{and} \\ I^v &:= \{i \mid \{z_i^l\} \text{ is uniformly bounded}, i = 1, \dots, m\}. \end{aligned}$$

Then we have

$$(44) \quad \Upsilon_i(\nu^l, z^l) \rightarrow -\infty \quad \forall i \in I^\infty$$

and

$$(45) \quad \Upsilon_i(\nu^l, z^l) \rightarrow \infty \quad \forall i \in I^{-\infty}.$$

For each  $l \geq 1$ , define  $h^l \in \mathbb{R}^m$  as follows:

$$h_i^l = \begin{cases} 0 & \text{if } i \in I^\infty \cup I^{-\infty}, \\ z_i^l & \text{if } i \in I^v, \end{cases} \quad i = 1, \dots, m.$$

Since, by part (ii) of Proposition 2.3, for any  $l \geq 1$ ,  $\Upsilon(\nu^l, \cdot)$  is a  $P_0$ -function, by further taking subsequences if necessary, we know that there exists  $i \in I^\infty \cup I^{-\infty}$  (note that  $h_j^l = z_j^l$  for all  $j \in I^v$  and  $l \geq 1$ ) such that

$$(z_i^l - h_i^l)(\Upsilon_i(\nu^l, z^l) - \Upsilon_i(\nu^l, h^l)) \geq 0 \quad \forall l \geq 1,$$

which is impossible in view of (44), (45), and the fact that  $\{\Upsilon(\nu^l, h^l)\}$  is bounded (note that  $\Upsilon$  is globally Lipschitz continuous). This shows that for any  $v \geq 0$ ,  $L_v$  is bounded; i.e.,

$$\{y \in \mathbb{R}^m \mid \|G(\varepsilon, y)\| \leq v, \varepsilon \in [\varepsilon', \hat{\varepsilon}]\}$$

is bounded. This implies that  $\{(\varepsilon^k, y^k)\}$  is bounded. Thus,  $\{(\varepsilon^k, y^k)\}$  has at least one accumulation point, which is a solution of  $E(\varepsilon, y) = 0$ , contradicting  $\bar{\varphi} > 0$ . Therefore,  $\bar{\varphi} = 0$ .

Suppose that the Slater condition (10) holds. Then from Proposition 2.1 we know that the solution set of the dual problem is nonempty and compact. Thus,  $E(\varepsilon, y) = 0$  also has a nonempty and compact solution set. Since part (ii) of Proposition 2.3 implies that  $E$  is a  $P_0$ -function, the boundedness of  $\{(\varepsilon^k, y^k)\}$  follows directly from [39, Theorem 2.5].  $\square$

Assume that the Slater condition (10) holds. Let  $(\bar{\varepsilon}, \bar{y})$  be an accumulation point of the infinite sequence  $\{(\varepsilon^k, y^k)\}$  generated by Algorithm 3.1. Then, by Theorem 4.1, we know that  $\bar{\varepsilon} = 0$  and  $F(\bar{y}) = 0$ ; i.e.,  $\bar{y} \in \mathcal{Q}^* = \mathbb{R}^p \times \mathbb{R}_+^q$  is an optimal solution to the dual problem (8). Let  $\bar{X} := \Pi_{\mathcal{S}_+^n}(C + \mathcal{A}^*\bar{y})$ . By Proposition 2.1 we know that  $\bar{X} \in \mathcal{S}_+^n$  is the unique optimal solution to problem (1).

For quadratic convergence of Algorithm 3.1, we need the concept of constraint nondegeneracy initiated by Robinson [40] and extensively developed by Bonnans and Shapiro [6]. This concept is a generalization of the well-known linear independence constraint qualification used in nonlinear programming. For a given closed  $K \in \mathcal{X}$ , a finite dimensional real Hilbert space, as in convex analysis [41] we use  $T_K(x)$  to denote the tangent cone of  $K$  at  $x \in K$ . The largest linear space contained in  $T_K(x)$  is denoted by  $\text{lin}(T_K(x))$ . Let  $\mathcal{I}$  be the identity mapping from  $\mathcal{S}^n$  to  $\mathcal{S}^n$ . Then the constraint nondegeneracy is said to hold at  $\bar{X}$  if

$$(46) \quad \begin{pmatrix} \mathcal{A} \\ \mathcal{I} \end{pmatrix} \mathcal{S}^n + \begin{pmatrix} \text{lin}(T_{\mathcal{Q}}(\mathcal{A}(\bar{X}) - b)) \\ \text{lin}(T_{\mathcal{S}_+^n}(\bar{X})) \end{pmatrix} = \begin{pmatrix} \mathbb{R}^m \\ \mathcal{S}^n \end{pmatrix},$$

where  $\mathcal{Q} = \{0\}^p \times \mathbb{R}_+^q$ . Note that the constraint nondegenerate condition (46) is called the primal nondegeneracy in [3].

Let  $\text{Ind}(\bar{X})$  denote the index set of active constraints at  $\bar{X}$ ,

$$\text{Ind}(\bar{X}) := \{i \mid \langle A_i, \bar{X} \rangle = b_i, i = p + 1, \dots, m\},$$

and  $s$  be the number of elements in  $\text{Ind}(\bar{X})$ . Without loss of generality, we assume that

$$\text{Ind}(\bar{X}) = \{p + 1, \dots, p + s\}.$$

Define  $\widehat{\mathcal{A}} : \mathcal{S}^n \rightarrow \mathbb{R}^{p+s}$  by

$$(47) \quad \widehat{\mathcal{A}}(X) := \begin{bmatrix} \langle A_1, X \rangle \\ \vdots \\ \langle A_{p+s}, X \rangle \end{bmatrix}, \quad X \in \mathcal{S}^n,$$

and the adjoint of  $\widehat{\mathcal{A}}$  is denoted by  $\widehat{\mathcal{A}}^*$ .

LEMMA 4.2. *Let  $X := C + \mathcal{A}^*\bar{y}$  have the spectral decomposition as in (15). Then the constraint nondegenerate condition (46) holds at  $\bar{X}$  if and only if for any  $h \in \mathbb{R}^{p+s}$ ,*

$$(48) \quad P_\alpha^T \widehat{\mathcal{A}}^* h = 0 \iff h = 0.$$

*Proof.* Since the linearity space  $\text{lin}(T_{\mathcal{Q}}(\mathcal{A}(\bar{X}) - b))$  in (46) can be computed directly as

$$(49) \quad \text{lin}(T_{\mathcal{Q}}(\mathcal{A}(\bar{X}) - b)) = \{h \in \mathbb{R}^m \mid h_i = 0, i = 1, \dots, p, i \in \text{Ind}(\bar{X})\},$$

we can see that (46) is reduced to

$$\begin{pmatrix} \widehat{\mathcal{A}} \\ \mathcal{I} \end{pmatrix} \mathcal{S}^n + \begin{pmatrix} \{0\}^{p+s} \\ \text{lin}(T_{\mathcal{S}_+^n}(\overline{X})) \end{pmatrix} = \begin{pmatrix} \mathbb{R}^{p+s} \\ \mathcal{S}^n \end{pmatrix},$$

which is equivalent to

$$(50) \quad \widehat{\mathcal{A}}(\text{lin } T_{\mathcal{S}_+^n}(\overline{X})) = \mathbb{R}^{p+s}.$$

Note that

$$\overline{X} = \Pi_{\mathcal{S}_+^n}(X) = P \text{diag}(\max(0, \lambda_1), \dots, \max(0, \lambda_n)) P^T,$$

the tangent cone  $T_{\mathcal{S}_+^n}(\overline{X})$ , which was first characterized by Arnold [5], takes the form

$$T_{\mathcal{S}_+^n}(\overline{X}) = \{B \in \mathcal{S}^n \mid [P_\beta \ P_\gamma]^T B [P_\beta \ P_\gamma] \succeq 0\}.$$

Consequently,

$$(51) \quad \text{lin}(T_{\mathcal{S}_+^n}(\overline{X})) = \{B \in \mathcal{S}^n \mid P_\beta^T B P_\beta = 0, P_\beta^T B P_\gamma = 0, P_\gamma^T B P_\gamma = 0\}.$$

Thus, from (50), the constraint nondegeneracy condition (46) holds if and only if (48) holds.  $\square$

LEMMA 4.3. *Let  $\Phi : \mathbb{R} \times \mathcal{S}^n \rightarrow \mathcal{S}^n$  be defined by (18). Assume that the constraint nondegeneracy (46) holds at  $\overline{X}$ . Then for any  $V \in \partial_B \Phi(0, \overline{X})$  we have*

$$(52) \quad \langle h, \widehat{\mathcal{A}}V(0, \widehat{\mathcal{A}}^*h) \rangle > 0 \quad \forall 0 \neq h \in \mathbb{R}^{p+s}.$$

*Proof.* Let  $V \in \partial_B \Phi(0, \overline{X})$ . Suppose that there exists  $0 \neq h \in \mathbb{R}^{p+s}$  such that (52) fails to hold, i.e.,

$$\langle h, \widehat{\mathcal{A}}V(0, \widehat{\mathcal{A}}^*h) \rangle \leq 0.$$

Denote  $H := \widehat{\mathcal{A}}^*h$ . Then, by Proposition 2.2, there exists  $V_{|\beta|} \in \partial_B \Phi_{|\beta|}(0, 0)$  such that

$$V(0, H) = P \begin{bmatrix} P_\alpha^T H P_\alpha & P_\alpha^T H P_\beta & U_{\alpha\gamma} \circ (P_\alpha^T H P_\gamma) \\ (P_\alpha^T H P_\beta)^T & V_{|\beta|}(0, P_\beta^T H P_\beta) & 0 \\ (P_\alpha^T H P_\gamma)^T \circ U_{\alpha\gamma}^T & 0 & 0 \end{bmatrix} P^T,$$

where  $U \in \mathcal{S}^n$  is defined by (23). Since  $\langle P_\beta^T H P_\beta, V_{|\beta|}(0, P_\beta^T H P_\beta) \rangle \geq 0$  and  $\langle h, \widehat{\mathcal{A}}V(0, \widehat{\mathcal{A}}^*h) \rangle \leq 0$ , we obtain from  $\langle h, \widehat{\mathcal{A}}V(0, \widehat{\mathcal{A}}^*h) \rangle = \langle H, V(0, H) \rangle$  that

$$P_\alpha^T H P_\alpha = 0, \quad P_\alpha^T H P_\beta = 0, \quad \text{and } P_\alpha^T H P_\gamma = 0,$$

i.e.,

$$P_\alpha^T H = P_\alpha^T \widehat{\mathcal{A}}^*h = 0.$$

On the other hand, since the constraint nondegeneracy (46) holds at  $\overline{X}$ , from (48) we know that  $h = 0$ . This contradiction shows that for any  $V \in \partial_B \Phi(0, X)$ , (52) holds.  $\square$



PROPOSITION 4.4. Let  $\Upsilon : \mathbb{R} \times \mathbb{R}^m \rightarrow \mathbb{R}^m$  be defined by (25). Assume that the constraint nondegeneracy (46) holds at  $\bar{X}$ . Then for any  $W \in \partial_B \Upsilon(0, \bar{y})$  we have

$$(53) \quad \max_i h_i(W(0, h))_i > 0 \quad \forall 0 \neq h \in \mathbb{R}^m.$$

*Proof.* Let  $W \in \partial_B \Upsilon(0, \bar{y})$ . Suppose that there exists  $0 \neq h \in \mathbb{R}^m$  such that (53) does not hold, i.e.,

$$(54) \quad \max_i h_i(W(0, h))_i \leq 0.$$

Then from part (iv) of Proposition 2.3 we know that there exist  $D \in \partial_B \psi(0, \bar{z})$  and  $V \in \partial_B \Phi(0, \bar{X})$  such that

$$(55) \quad W(0, h) = h - D(0, h - \mathcal{A}V(0, \mathcal{A}^*h)) = h - D(0, h) + D(0, \mathcal{A}V(0, \mathcal{A}^*h)),$$

where  $\bar{z} := \bar{y} - \nabla\theta(\bar{y}) = \bar{y} - (\mathcal{A}\Phi(0, \bar{X}) - b)$ . By simple calculations, we can see that there exists a nonnegative vector  $d \in \mathbb{R}^m$  satisfying

$$d_i = \begin{cases} 1 & \text{if } 1 \leq i \leq p, \\ \in [0, 1] & \text{if } p + 1 \leq i \leq p + s, \\ 0 & \text{if } p + s + 1 \leq i \leq m \end{cases}$$

such that for any  $y \in \mathbb{R}^m$ ,

$$(D(0, y))_i = d_i y_i, \quad i = 1, \dots, m.$$

Thus, we obtain from (55) and (54) that

$$\begin{cases} h_i(\mathcal{A}V(0, \mathcal{A}^*h))_i \leq 0 & \text{if } 1 \leq i \leq p, \\ h_i(\mathcal{A}V(0, \mathcal{A}^*h))_i \leq 0 \text{ or } h_i = 0 & \text{if } p + 1 \leq i \leq p + s, \\ h_i = 0 & \text{if } p + s + 1 \leq i \leq m, \end{cases}$$

which implies

$$\langle h, \mathcal{A}V(0, \mathcal{A}^*h) \rangle = \langle \hat{h}, \hat{\mathcal{A}}V(0, \hat{\mathcal{A}}^*\hat{h}) \rangle \leq 0,$$

where  $0 \neq \hat{h} \in \mathbb{R}^{p+s}$  is defined by  $\hat{h}_i = h_i, i = 1, \dots, p + s$ . This, however, contradicts (52) in Lemma 4.3. This contradiction shows that (53) holds.  $\square$

THEOREM 4.5. Let  $(\bar{\varepsilon}, \bar{y})$  be an accumulation point of the infinite sequence  $\{(\varepsilon^k, y^k)\}$  generated by Algorithm 3.1. Assume that the constraint nondegeneracy (46) holds at  $\bar{X}$ . Then the whole sequence  $\{(\varepsilon^k, y^k)\}$  converges to  $(\bar{\varepsilon}, \bar{y})$  quadratically, i.e.,

$$(56) \quad \|(\varepsilon^{k+1} - \bar{\varepsilon}, y^{k+1} - \bar{y})\| = O(\|(\varepsilon^k - \bar{\varepsilon}, y^k - \bar{y})\|^2).$$

*Proof.* In order to apply Theorem 3.6 to obtain the quadratic convergence of  $\{(\varepsilon^k, y^k)\}$ , we need only to check that  $E$  is strongly semismooth at  $(\bar{\varepsilon}, \bar{y})$  and that all  $V \in \partial_B E(\bar{\varepsilon}, \bar{y})$  are nonsingular.

The strong semismoothness of  $E$  at  $(\bar{\varepsilon}, \bar{y})$  follows directly from part (iii) of Proposition 2.3 and the fact that the modulus function  $|\cdot|$  is strongly semismooth everywhere on  $\mathbb{R}$ . The nonsingularity of all matrices in  $\partial_B E(\bar{\varepsilon}, \bar{y})$  can be proven as follows.

Let  $V \in \partial_B E(\bar{x}, \bar{y})$  be arbitrarily chosen. From Proposition 4.4 and the definition of  $E$ , we know that for any  $0 \neq d \in \mathbb{R}^{m+1}$ ,

$$\max_i d_i (Vd)_i > 0,$$

which, by [14, Theorem 3.3.4], implies that  $V$  is a  $P$ -matrix and so nonsingular. Then the proof is completed.  $\square$

Theorem 4.5 says that Algorithm 3.1 can achieve quadratic convergence under the assumption that the constraint nondegenerate condition (46) holds at  $\bar{X}$ . Next, we shall discuss this assumption by considering the following special LSSDP:

$$(57) \quad \begin{aligned} \min \quad & \frac{1}{2} \|X - C\|^2 \\ \text{s.t.} \quad & X_{ij} = e_{ij}, \quad (i, j) \in \mathcal{B}_e, \\ & X_{ij} \geq l_{ij}, \quad (i, j) \in \mathcal{B}_l, \\ & X_{ij} \leq u_{ij}, \quad (i, j) \in \mathcal{B}_u, \\ & X \in \mathcal{S}_+^n, \end{aligned}$$

where  $\mathcal{B}_e$ ,  $\mathcal{B}_l$ , and  $\mathcal{B}_u$  are three index subsets of  $\{(i, j) \mid 1 \leq i \leq j \leq n\}$  satisfying  $\mathcal{B}_e \cap \mathcal{B}_l = \emptyset$ ,  $\mathcal{B}_e \cap \mathcal{B}_u = \emptyset$ , and  $l_{ij} < u_{ij}$  for any  $(i, j) \in \mathcal{B}_l \cap \mathcal{B}_u$ . Denote the cardinalities of  $\mathcal{B}_e$ ,  $\mathcal{B}_l$ , and  $\mathcal{B}_u$  by  $p$ ,  $q_l$ , and  $q_u$ , respectively. Let  $m := p + q_l + q_u$ . For any  $(i, j) \in \{1, \dots, n\} \times \{1, \dots, n\}$ , define  $\mathcal{E}^{ij} \in \mathbb{R}^{n \times n}$  by

$$(\mathcal{E}^{ij})_{st} := \begin{cases} 1 & \text{if } (s, t) = (i, j), \\ 0 & \text{otherwise,} \end{cases} \quad s, t = 1, \dots, n.$$

Thus, problem (57) can be written as a special case of (1) with

$$(58) \quad \mathcal{A}(X) := \begin{bmatrix} \{\langle A^{ij}, X \rangle\}_{(i,j) \in \mathcal{B}_e} \\ \{\langle A^{ij}, X \rangle\}_{(i,j) \in \mathcal{B}_l} \\ -\{\langle A^{ij}, X \rangle\}_{(i,j) \in \mathcal{B}_u} \end{bmatrix}, \quad X \in \mathcal{S}^n,$$

and

$$b := \begin{pmatrix} \{e_{ij}\}_{(i,j) \in \mathcal{B}_e} \\ \{l_{ij}\}_{(i,j) \in \mathcal{B}_l} \\ -\{u_{ij}\}_{(i,j) \in \mathcal{B}_u} \end{pmatrix},$$

where  $A^{ij} := \frac{1}{2}(\mathcal{E}^{ij} + \mathcal{E}^{ji})$ . Then its dual problem takes the same form as (8) with  $q := q_l + q_u$ . The index set  $\text{Ind}(\bar{X})$  of active constraints at  $\bar{X}$  now becomes

$$\text{Ind}(\bar{X}) = \widehat{\mathcal{B}}_l \cup \widehat{\mathcal{B}}_u,$$

where

$$\widehat{\mathcal{B}}_l := \{(i, j) \in \mathcal{B}_l \mid \langle A^{ij}, \bar{X} \rangle = l_{ij}\} \quad \text{and} \quad \widehat{\mathcal{B}}_u := \{(i, j) \in \mathcal{B}_u \mid \langle A^{ij}, \bar{X} \rangle = u_{ij}\}.$$

Let  $s$  be the cardinality of  $\text{Ind}(\bar{X})$ . Then the mapping  $\widehat{\mathcal{A}} : \mathcal{S}^n \rightarrow \mathbb{R}^{p+s}$  defined by (47) takes the form

$$\widehat{\mathcal{A}}(X) := \begin{bmatrix} \{\langle A^{ij}, X \rangle\}_{(i,j) \in \mathcal{B}_e} \\ \{\langle A^{ij}, X \rangle\}_{(i,j) \in \widehat{\mathcal{B}}_l} \\ -\{\langle A^{ij}, X \rangle\}_{(i,j) \in \widehat{\mathcal{B}}_u} \end{bmatrix}.$$

Recall that the constraint nondegenerate condition (46) holds at  $\overline{X}$  if and only if for any  $h \in \mathbb{R}^{p+s}$ , (48) holds. A particular case for (48) to hold is when  $\mathcal{B}_e = \{(i, i) \mid i = 1, \dots, n\}$ ,  $\mathcal{B}_l \cup \mathcal{B}_u = \emptyset$ , and  $b > 0$  [34, 35]. Furthermore, if  $\mathcal{B}_e$  has a band structure, (48) also holds as long as the corresponding band of the given matrix  $C$  is positive definite [35]. In general, the equivalent constraint nondegenerate condition (48) may fail to hold for problem (57). In [33], Qi establishes an interesting connection between the constraint nondegeneracy and the positive semidefinite matrix completions on chordal graphs.

**5. Numerical results.** In this section, we report our numerical experiments conducted for testing the efficiency of Algorithm 3.1. The main task of Algorithm 3.1 for solving the LSSDP (1), at the  $k$ th iterate, is to solve the linear system (27) with  $E(\cdot)$  being defined by (43). In numerical implementation, we first obtain  $\Delta\varepsilon^k = -\varepsilon^k + \zeta_k \hat{\varepsilon}$  by (27) and then apply the BiCGStab iterative solver of van der Vorst [49] to the resulting linear system

$$(59) \quad G'_y(\varepsilon^k, y^k)\Delta y^k = -G(\varepsilon^k, y^k) - G'_\varepsilon(\varepsilon^k, y^k)\Delta\varepsilon^k$$

to obtain  $\Delta y^k$  such that it satisfies (28). The cost of using the BiCGStab iterative solver to compute  $\Delta y^k$  will be analyzed next.

For the sake of convenience, in subsequent analysis we suppress the superscript  $k$ . By noting that  $G(\varepsilon, y)$  and  $\Upsilon(\varepsilon, y)$  are defined by (42) and (25), respectively, we obtain that

$$(60) \quad \begin{aligned} G'_y(\varepsilon, y)\Delta y &= \Upsilon'_y(\varepsilon, y)\Delta y + \kappa\varepsilon\Delta y \\ &= \Delta y - \psi'_z(\varepsilon, z)(\Delta y - \mathcal{A}\Phi'_X(\varepsilon, X)(\mathcal{A}^*\Delta y)) + \kappa\varepsilon\Delta y \\ &= \Delta y - \psi'_z(\varepsilon, z)\Delta y + \psi'_z(\varepsilon, z)(\mathcal{A}\Phi'_X(\varepsilon, X)(\mathcal{A}^*\Delta y)) + \kappa\varepsilon\Delta y, \end{aligned}$$

where  $z := y - (\mathcal{A}\Phi(\varepsilon, X) - b)$ ,  $X := C + \mathcal{A}^*y$ , and  $\mathcal{A}^*$  is given by (9). Let  $X$  have the spectral decomposition (15). Then, by (19),

$$(61) \quad \Phi'_X(\varepsilon, X)(\mathcal{A}^*\Delta y) = P[\Omega(\varepsilon, \lambda) \circ (P^T \mathcal{A}^* \Delta y P)]P^T,$$

where  $\Omega(\varepsilon, \lambda)$  is given by (20).

From (60) and (61), we know that in order to compute the  $m$  by  $m$  matrix  $G'_y(\varepsilon, y)$  one needs  $O(mn^3)$  flops ( $m$  columns and  $O(n^3)$  flops for entries in each column). This implies that it is impractical to use direct methods to solve the linear system (59) even when  $n$  and  $m$  are not large, say,  $n = 1,000$  and  $m = 1,000$ . Given the fact that the matrix  $G'_y(\varepsilon, y)$  is nonsymmetric when problem (1) has inequality constraints, i.e.,  $q \neq 0$ , it is natural to choose the BiCGStab as our iterative solver for solving the linear system (59).

In order to speed up the convergence of the BiCGStab iterative solver, in the following we shall introduce a diagonal preconditioner. Define the vector  $d \in \mathbb{R}^m$  as the diagonal part of the coefficient matrix  $G'_y(\varepsilon, y)$ , i.e.,  $d := \text{diag}(G'_y(\varepsilon, y))$ . Then from (60) and (61) we know for each  $l \in \{1, \dots, m\}$  that

$$(62) \quad d_l = [G'_y(\varepsilon, y)]_{ll} = [G'_y(\varepsilon, y)I_l]_l = 1 - \phi'_{z_l}(\varepsilon, z_l) + \phi'_{z_l}(\varepsilon, z_l)w_l + \kappa\varepsilon,$$

where  $I_l \in \mathbb{R}^m$  denotes the  $l$ th column of the identity matrix  $I$  and  $w_l$  is defined by

$$\begin{aligned}
 w_l &:= [\mathcal{A}\Phi'_X(\varepsilon, X)(\mathcal{A}^*I_l)]_l \\
 &= \langle I_l, \mathcal{A}\Phi'_X(\varepsilon, X)(\mathcal{A}^*I_l) \rangle \\
 &= \langle \mathcal{A}^*I_l, \Phi'_X(\varepsilon, X)(\mathcal{A}^*I_l) \rangle \\
 &= \langle \mathcal{A}^*I_l, P[\Omega(\varepsilon, \lambda) \circ (P^T \mathcal{A}^*I_l P)]P^T \rangle \\
 (63) \quad &= \langle P^T A_l P, \Omega(\varepsilon, \lambda) \circ (P^T A_l P) \rangle.
 \end{aligned}$$

Let  $D \in \mathbb{R}^{m \times m}$  be defined by  $D := \text{diag}(d)$ . On the one hand, the diagonal matrix  $D$  is an obvious diagonal preconditioner for the linear system (59). On the other hand, from (62) and (63), we know that for a large  $m$  it is expensive to compute the diagonal matrix  $D$  because its computation needs  $O(mn^3)$  flops. Of course, the cost for computing  $D$  can be reduced if most of the matrices  $A_l$  are sparse. For instance, in problem (57), for each  $l \in \{1, \dots, m\}$ , the matrix  $A_l$  takes the form

$$A_l = \frac{1}{2}(\mathcal{E}^{i_l j_l} + \mathcal{E}^{j_l i_l}),$$

where

$$(i_l, j_l) \in \begin{cases} \mathcal{B}_e & \text{if } l = 1, \dots, p, \\ \mathcal{B}_l & \text{if } l = p + 1, \dots, p + q_l, \\ \mathcal{B}_u & \text{if } l = p + q_l + 1, \dots, m. \end{cases}$$

By taking into account such a special structure of  $A_l$ ,  $w_l$  can be further simplified as follows:

$$\begin{aligned}
 w_l &= \langle P^T A_l P, \Omega(\varepsilon, \lambda) \circ (P^T A_l P) \rangle \\
 (64) \quad &= \frac{1}{2} [a_{i_l}^2 \Omega(\varepsilon, \lambda)(a_{j_l}^2)^T + (a_{i_l} \circ a_{j_l})\Omega(\varepsilon, \lambda)(a_{i_l} \circ a_{j_l})^T], \quad l = 1, \dots, m,
 \end{aligned}$$

where “ $\circ$ ” denotes the Hadamard product of two vectors,  $a_i$  is the  $i$ th row of  $P$ , and  $a_i^2 := a_i \circ a_i$ ,  $i = 1, \dots, n$ . Thus, in this case the diagonal matrix  $D$  can be computed directly from (62) and (64) with a reduced cost of  $O(mn^2)$  flops. However, when  $m$  is much larger than  $n$ , say,  $m = O(n^2)$ , this cost is still too expensive. In our numerical implementation, we use an estimated diagonal matrix of  $D$  as our diagonal preconditioner. This estimated diagonal matrix can be obtained in  $O(n^3)$  flops as follows.

Define  $\tilde{w} \in \mathbb{R}^m$  and  $\tilde{d} \in \mathbb{R}^m$  by

$$(65) \quad \tilde{w}_l := \frac{1}{2} a_{i_l}^2 \Omega(\varepsilon, \lambda)(a_{j_l}^2)^T, \quad l = 1, \dots, m,$$

and

$$(66) \quad \tilde{d}_l := 1 - \phi'_{z_l}(\varepsilon, z_l) + \phi'_{z_l}(\varepsilon, z_l)\tilde{w}_l + \kappa\varepsilon, \quad l = 1, \dots, m,$$

respectively. Let  $\tilde{D} := \text{diag}(\tilde{d})$ . It is easy to see that computing the diagonal matrix  $\tilde{D}$  requires only  $O(n^3)$  flops, which are independent of  $m$ . We simply use the diagonal matrix  $\tilde{D}$  as our diagonal preconditioner. Numerical results further show that such a heuristically estimated preconditioner works pretty well.

*Remark 5.1.* For the general LSSDP problem (1), if all the matrices  $A_i$  are of rank 1, we can still apply the above technique to obtain a diagonal preconditioner with a cost of  $O(n^3)$ .

In our numerical experiments we compare our inexact smoothing Newton method, which is referred to as **Smoothing** in our numerical results, with the following methods for solving the LSSDP with simple constraints (57):

- (i) The projected gradient method (PGM) of Boyd and Xiao [10]. At the  $k$ th step, the PGM computes

$$y^{k+1} := \Pi_{\mathcal{Q}^*}(y^k - \rho \nabla \theta(y^k)),$$

where  $\rho \in (0, 2)$ . Note that in this case the Lipschitz modulus of  $\nabla \theta(y)$  is one. We take  $\rho = 1.8$  as it performs better than  $\rho = 1$ . The maximum number of iterations is set as 5,000, and the stopping criterion is

$$Res_k := \|\nabla \theta(y^k)\| \leq 10^{-5}.$$

Note that when there are no inequality constraints, i.e.,  $m = p$ , the PGM reduces to the gradient method (GM) of Higham [25].

- (ii) The BFGS-SQP method of Chen [12] and Malick [29]. In [29], Malick suggested using a BFGS type method to solve the dual problem (8). Here we adopt a BFGS-SQP approach introduced by Chen [12] for solving a more general  $LC^1$  problem. For the dual problem (8), at the  $k$ th step, the BFGS-SQP method solves the following quadratic program:

$$(67) \quad \begin{aligned} \min \quad & \langle \nabla \theta(y^k), \Delta y^k \rangle + \frac{1}{2} \langle \Delta y^k, (B_k + \epsilon_k I) \Delta y^k \rangle \\ \text{s.t.} \quad & y^k + \Delta y^k \in \mathcal{Q}^*. \end{aligned}$$

We use Algorithm 3.1 to compute the subproblem (67). The maximum number of iterations is set as 2,000, and the stopping criterion is the same as in the PGM. For cases with equality constraints only, instead of using our own code, we run the MATLAB package *SDLS*, which is an implementation of the BFGS method for the dual problem and is written by Henrion and Malick [24].

- (iii) The inexact semismooth Newton method of Qi and Sun [34], which is referred to as **Semismooth** in our numerical results. This algorithm is particularly designed for the nearest correlation matrix problem. In order to improve its performance and for comparison purposes, we introduced a diagonal preconditioner by using the same technique as in our inexact smoothing Newton method. The stopping criterion is

$$Res_k := \|\nabla \theta(y^k)\| \leq 10^{-6}.$$

- (iv) The inexact interior point method of Toh, Tütüncü, and Todd [47] for solving the  $W$ -weighted nearest correlation matrix problem (7). This inexact interior point method, referred to as IP-NCM here, uses a preconditioned symmetric quasi-minimal residual iterative method as the iterative solver and can solve equality constrained cases when applied to problem (57). The stopping criterion is  $Res_k \leq 10^{-7}$ . Note that the “*Res*” defined in the IP-NCM method is the relative residue rather than the absolute residue of the resulting Karush–Kuhn–Tucker system of problem (57).

The stopping criterion chosen for Algorithm 3.1 is

$$Res_k := \|E(\varepsilon^k, y^k)\| \leq 10^{-6}.$$

The maximum number of BiCGStab steps at each iteration is set as 200. The other parameters used in Algorithm 3.1 are set as  $r = 0.2$ ,  $\hat{\varepsilon} = 0.05$ ,  $\eta = 0.5$ ,  $\rho = 0.5$ ,  $\sigma = 0.5 \times 10^{-6}$ ,  $\tau = 0.01$ ,  $\hat{\tau} = 0.5$ , and  $\kappa = 0.01$ .

*Remark 5.2.* For the BFGS and IP-NCM methods, memory is a big issue since they both need to compute and store an  $m$  by  $m$  matrix at each iteration.

We implemented all algorithms in MATLAB 7.1 running on a PC Intel Pentium IV of 2.40 GHz CPU and 512 MB of RAM. The testing examples are given below.

*Example 5.1.* The matrix  $C$  is the  $387 \times 387$  1-day correlation matrix (as of June 15, 2006) from the lagged datasets of RiskMetrics.<sup>1</sup> For test purposes, we perturb  $C$  to

$$C := (1 - \alpha)C + \alpha R,$$

where  $\alpha \in (0, 1)$  and  $R$  is a randomly generated symmetric matrix with entries in  $[-1, 1]$ . The MATLAB code for generating the random matrix  $R$  is as follows: `R = 2.0*rand(387,387)-ones(387,387); R = triu(R)+triu(R,1)'; for i=1:387; R(i,i) = 1; end.` Here we take  $\alpha = 0.1$  and the index sets

$$\mathcal{B}_e := \{(i, i) \mid i = 1, \dots, 387\}, \quad \mathcal{B}_l \cup \mathcal{B}_u = \emptyset, \quad \text{and } e_{ii} = 1 \text{ for } (i, i) \in \mathcal{B}_e.$$

Note that this example corresponds exactly to the nearest correlation matrix problem.

*Example 5.2.* All the data are the same as in Example 5.1 except that  $e_{ii} \in [0, 1]$  for  $(i, i) \in \mathcal{B}_e$  are randomly generated. This example corresponds to the  $W$ -weighted nearest correlation problem (7) when the weight matrix  $W$  is a randomly generated diagonal matrix. It can also come from the local correlation stress testing [35].

*Example 5.3.* The matrix  $C$  is a randomly generated  $n \times n$  symmetric matrix with entries in  $[-1, 1]$  : `C = 2.0*rand(n,n)-ones(n,n); C = triu(C)+triu(C,1)'; for i=1:n; C(i,i) = 1; end.` The index sets

$$\mathcal{B}_e = \{(i, i) \mid i = 1, \dots, n\} \text{ and } \mathcal{B}_l = \mathcal{B}_u = \{(i, i + j) \mid i = 1, \dots, n - j, j = 1, 2\}.$$

We take  $e_{ii} = 1$  for  $(i, i) \in \mathcal{B}_e$ ,  $l_{ij} = -0.1$  for  $(i, j) \in \mathcal{B}_l$ , and  $u_{ij} = 0.1$  for  $(i, j) \in \mathcal{B}_u$  and test for  $n = 500, 1,000$ , and  $2,000$ , respectively. Note that in this example the lower and upper bounds defined fixed elements in active set form a chordal graph, and therefore, the constraint nondegeneracy (46) holds from [33, Theorem 3.7].

*Example 5.4.* Let  $n = 387$ . The data  $C$ ,  $\mathcal{B}_e$ , and  $e_{ii}$  are the same as in Example 5.1; i.e.,  $C$  is the perturbed  $387 \times 387$  correlation matrix and  $e_{ii} = 1$  for  $(i, i) \in \mathcal{B}_e = \{(i, i) \mid i = 1, \dots, n\}$ . The index sets  $\mathcal{B}_l, \mathcal{B}_u \subset \{(i, j) \mid 1 \leq i < j \leq n\}$  consist of the indices of  $\min(\hat{n}_r, n - i)$  randomly generated elements at the  $i$ th row of  $X$ ,  $i = 1, \dots, n$ , with  $\hat{n}_r$  taking the following values: (a)  $\hat{n}_r = 1$  ( $q_l = q_u = 386$ ); (b)  $\hat{n}_r = 2$  ( $q_l = q_u = 771$ ); (c)  $\hat{n}_r = 5$  ( $q_l = q_u = 1,920$ ); (d)  $\hat{n}_r = 10$  ( $q_l = q_u = 3,815$ ); (e)  $\hat{n}_r = 20$  ( $q_l = q_u = 7,530$ ). We take  $l_{ij} = -0.1$  for  $(i, j) \in \mathcal{B}_l$  and  $u_{ij} = 0.1$  for  $(i, j) \in \mathcal{B}_u$ .

*Example 5.5.* All the data are the same as in Example 5.4 except that

$$e_{ii} = \alpha + (1 - \alpha)\omega, \quad (i, i) \in \mathcal{B}_e,$$

where  $\alpha = 0.1$  and  $\omega$  is a randomly generated number in  $[0, 1]$ .

*Example 5.6.* The matrix  $C$  is the same as in Example 5.3, i.e., a randomly generated  $n \times n$  symmetric matrix with entries in  $[-1, 1]$ . The index sets  $\mathcal{B}_e, \mathcal{B}_l$ , and  $\mathcal{B}_u$  are the same as in Example 5.4 with  $\hat{n}_r = 1, 5$ , and  $10$ . We consider the following two cases: (a)  $e_{ii} = 1$ ,  $(i, i) \in \mathcal{B}_e$  and (b)  $e_{ii} = \alpha + (1 - \alpha)\omega$ ,  $(i, i) \in \mathcal{B}_e$ , where  $\alpha = 0.1$  and  $\omega$  is a randomly generated number in  $[0, 1]$ . We test for  $n = 500, 1,000$ , and  $2,000$ , respectively.

<sup>1</sup><http://www.riskmetrics.com/>

*Example 5.7.* All the data are the same as in Example 5.4 except that  $l_{ij} = 0.5$  for  $(i, j) \in \mathcal{B}_l$  and  $u_{ij} = 0.8$  for  $(i, j) \in \mathcal{B}_u$ .

*Example 5.8.* All the data are the same as in Example 5.7 except that  $l_{ij} = 0.8$  for  $(i, j) \in \mathcal{B}_l$  and  $u_{ij} = 0.9$  for  $(i, j) \in \mathcal{B}_u$ .

*Remark 5.3.* In order to make the tested examples general and avoid any special structures of the index sets, we randomly choose the elements which are to be fixed or with lower and (or) upper bound in all the examples above except for Example 5.3 which is particularly designed to guarantee the constraint nondegeneracy assumption.

Our numerical results are reported in Tables 1–5. “Iter” and “Res” stand for the number of total iterations and the residue at the final iterate of an algorithm, respectively. “\*” means that an algorithm reaches the set maximum number of iterations before the accuracy is achieved, while “out of memory” means that our computer runs out of memory. The maximum cputime set for each algorithm is 10 hours. So “> 10 hrs” indicates that an algorithm is terminated after 10 hours.

From the numerical results reported in Tables 1–5, we can see that for cases with equality constraints only (Table 1), Newton-type methods (**Semismooth**, **IP-NCM**, and **Smoothing**) can achieve higher accuracy than the gradient based methods (**GM** and **BFGS**). The **Smoothing** method is comparable to the well-tested **Semismooth** method

TABLE 1

|            | Example 5.1 |         |        | Example 5.2 |         |        |
|------------|-------------|---------|--------|-------------|---------|--------|
| Method     | Iter        | cputime | Res    | Iter        | cputime | Res    |
| GM         | 41          | 0:34    | 8.0e-6 | 5000*       | 58:49   | 4.4e-2 |
| BFGS       | 17          | 0:16    | 8.4e-6 | 1203        | 14:20   | 9.1e-6 |
| Semismooth | 5           | 0:07    | 2.3e-7 | 12          | 0:15    | 3.8e-8 |
| IP-NCM     | 11          | 0:54    | 8.0e-9 | 19          | 2:01    | 1.3e-8 |
| Smoothing  | 5           | 0:07    | 4.6e-7 | 12          | 0:14    | 3.3e-7 |

TABLE 2

| Example 5.3 | n=500 |         |        | n=1000        |         |        | n=2000        |         |        |
|-------------|-------|---------|--------|---------------|---------|--------|---------------|---------|--------|
| Method      | Iter  | cputime | Res    | Iter          | cputime | Res    | Iter          | cputime | Res    |
| PGM         | 108   | 3:12    | 9.7e-6 | 146           | 31:27   | 9.8e-6 | 197           | 5:07:04 | 9.8e-6 |
| BFGS-SQP    | 41    | 16:26   | 9.4e-6 | out of memory |         |        | out of memory |         |        |
| Smoothing   | 7     | 0:21    | 2.1e-8 | 8             | 2:50    | 1.0e-7 | 9             | 25:19   | 4.1e-7 |

TABLE 3

|           |      | Example 5.4   |         |        | Example 5.5   |         |        |
|-----------|------|---------------|---------|--------|---------------|---------|--------|
| Method    | Case | Iter          | cputime | Res    | Iter          | cputime | Res    |
| PGM       | (a)  | 993           | 12:13   | 9.9e-6 | 5000*         | 1:02:14 | 2.6e-5 |
|           | (b)  | 931           | 12:10   | 1.0e-5 | 5000*         | 1:03:23 | 2.7e-3 |
|           | (c)  | 895           | 12:07   | 1.0e-5 | 5000*         | 1:05:23 | 1.8e-4 |
|           | (d)  | 1018          | 13:42   | 1.0e-5 | 5000*         | 1:06:18 | 1.6e-4 |
|           | (e)  | 824           | 11:38   | 9.9e-6 | 5000*         | 1:06:33 | 5.3e-5 |
| BFGS-SQP  | (a)  | 130           | 7:21    | 9.8e-6 | 350           | 0:35:00 | 9.3e-6 |
|           | (b)  | 142           | 35:18   | 9.7e-6 | 450           | 3:37:53 | 1.0e-5 |
|           | (c)  | out of memory |         |        | out of memory |         |        |
|           | (d)  | --            |         |        | --            |         |        |
|           | (e)  | --            |         |        | --            |         |        |
| Smoothing | (a)  | 7             | 0:25    | 8.9e-7 | 11            | 0:37    | 4.7e-7 |
|           | (b)  | 8             | 0:24    | 1.0e-8 | 14            | 0:49    | 7.3e-8 |
|           | (c)  | 8             | 0:23    | 1.7e-8 | 16            | 0:52    | 5.2e-7 |
|           | (d)  | 9             | 0:32    | 7.2e-8 | 24            | 1:47    | 2.4e-7 |
|           | (e)  | 8             | 0:26    | 2.9e-7 | 22            | 1:36    | 2.7e-8 |

TABLE 4

| Example 5.6 | n=500 |              |               | n=1000  |        |               | n=2000  |        |               |         |        |
|-------------|-------|--------------|---------------|---------|--------|---------------|---------|--------|---------------|---------|--------|
| Method      | Case  | $ \hat{n}_r$ | Iter          | cputime | Res    | Iter          | cputime | Res    | Iter          | cputime | Res    |
| PGM         | (a)   | 1            | 104           | 3:25    | 9.2e-6 | 140           | 28:56   | 9.9e-6 | 191           | 4:40:57 | 9.6e-6 |
|             |       | 5            | 128           | 4:17    | 9.6e-6 | 165           | 35:30   | 9.7e-6 | 210           | 5:22:03 | 9.9e-6 |
|             |       | 10           | 151           | 4:55    | 9.6e-6 | 183           | 39:35   | 9.9e-6 | 227           | 5:49:26 | 9.7e-6 |
|             | (b)   | 1            | 936           | 26:20   | 9.9e-6 | 1160          | 3:57:58 | 1.0e-5 | 412           | >10 hrs | 3.0e-2 |
|             |       | 5            | 795           | 23:01   | 9.9e-6 | 1153          | 4:00:12 | 1.0e-5 | 396           | >10 hrs | 3.4e-2 |
|             |       | 10           | 947           | 27:43   | 1.0e-5 | 1205          | 4:11:30 | 1.0e-5 | 392           | >10 hrs | 3.3e-2 |
| BFGS-SQP    | (a)   | 1            | 41            | 4:15    | 9.1e-6 | 45            | 35:44   | 9.0e-6 | out of memory |         |        |
|             |       | 5            | out of memory |         |        | out of memory |         |        | ---           |         |        |
|             |       | 10           | ---           |         |        | ---           |         |        | ---           |         |        |
|             | (b)   | 1            | 124           | 17:30   | 9.6e-6 | 141           | 3:04:18 | 9.9e-6 | out of memory |         |        |
|             |       | 5            | out of memory |         |        | out of memory |         |        | ---           |         |        |
|             |       | 10           | ---           |         |        | ---           |         |        | ---           |         |        |
| Smoothing   | (a)   | 1            | 7             | 0:23    | 2.7e-8 | 8             | 2:42    | 6.3e-9 | 8             | 19:27   | 3.3e-8 |
|             |       | 5            | 7             | 0:24    | 6.7e-7 | 8             | 2:38    | 2.9e-7 | 8             | 18:53   | 2.4e-7 |
|             |       | 10           | 7             | 0:24    | 1.0e-6 | 8             | 2:47    | 1.8e-7 | 9             | 21:10   | 2.1e-8 |
|             | (b)   | 1            | 9             | 0:29    | 7.4e-7 | 10            | 3:22    | 3.9e-9 | 10            | 24:56   | 8.9e-9 |
|             |       | 5            | 9             | 0:26    | 2.3e-7 | 9             | 2:53    | 3.3e-7 | 10            | 22:38   | 6.5e-8 |
|             |       | 10           | 10            | 0:32    | 9.1e-9 | 11            | 4:01    | 3.4e-7 | 11            | 29:05   | 8.9e-9 |

TABLE 5

|           |      | Example 5.7   |         |        | Example 5.8   |         |        |
|-----------|------|---------------|---------|--------|---------------|---------|--------|
| Method    | Case | Iter          | cputime | Res    | Iter          | cputime | Res    |
| PGM       | (a)  | 1437          | 18:51   | 1.0e-5 | 5000*         | 1:02:22 | 1.3e-3 |
|           | (b)  | 1409          | 18:25   | 1.0e-5 | 5000*         | 1:05:55 | 4.0e-2 |
|           | (c)  | 1483          | 19:55   | 1.0e-5 | 5000*         | 1:07:46 | 2.4e-2 |
|           | (d)  | 1587          | 21:32   | 1.0e-5 | 5000*         | 1:08:35 | 7.1e-3 |
|           | (e)  | 1335          | 18:15   | 9.9e-6 | 5000*         | 1:04:22 | 4.0e-4 |
| BFGS-SQP  | (a)  | 173           | 11:59   | 9.6e-6 | 413           | 0:39:52 | 9.9e-6 |
|           | (b)  | 171           | 46:33   | 9.6e-6 | 737           | 7:30:13 | 9.6e-6 |
|           | (c)  | out of memory |         |        | out of memory |         |        |
|           | (d)  | ---           |         |        | ---           |         |        |
|           | (e)  | ---           |         |        | ---           |         |        |
| Smoothing | (a)  | 9             | 0:35    | 1.3e-8 | 11            | 0:58    | 8.8e-7 |
|           | (b)  | 9             | 0:30    | 3.8e-8 | 21            | 2:46    | 6.4e-7 |
|           | (c)  | 11            | 0:30    | 1.9e-7 | 33            | 4:19    | 4.2e-7 |
|           | (d)  | 11            | 0:33    | 7.0e-7 | 31            | 3:31    | 8.2e-8 |
|           | (e)  | 12            | 0:47    | 2.3e-7 | 35            | 6:00    | 9.8e-7 |

in terms of cputime. Both the **Smoothing** and **Semismooth** methods are faster (2–8 times) than the **IP-NCM** method for achieving similar accuracies. The performance of the **BFGS** method varies from case to case. It is more competitive than the **IP-NCM** method for the tested unweighted nearest correlation matrix problems, but it is much slower than the **IP-NCM** method for the weighted cases. The **GM** method is easy to implement, but it may converge extremely slow or not converge at all due to truncated numerical errors. For cases with both equality and inequality constraints (Tables 2–5), we have only three methods—**PGM**, **BFGS-SQP**, and **Smoothing**—to test. The performance of the **BFGS-SQP** method is very poor partially because it has to solve a quadratic programming problem at each iteration and needs much more memory than the **PGM** and **Smoothing** methods. In all tested examples with both equality and inequality constraints, the **Smoothing** method is much more efficient.

**6. Conclusions.** In this paper, we proposed a quadratically convergent inexact smoothing Newton method—Algorithm 3.1—for solving the LSSDP (1). Our



approach is to reformulate the problem as a smoothing-nonsmooth system of equations. Under mild conditions, we showed that the reformulated system enjoys a desirable nonsingularity property, which is vital for applying the BiCGStab iterative solver to the resulting smoothing Newton linear system. Our conducted numerical results<sup>2</sup> clearly demonstrated that Algorithm 3.1 is very efficient for solving the LSSDP with simple constraints (57), for which we introduced a simple diagonal preconditioner with a low cost.

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<sup>2</sup>The MATLAB code is available from <http://www.math.nus.edu.sg/~matsundf/>.

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